Notes On

HAHN-BANACH THEOREM

Extension Form

I start with a definition – that of a *sublinear functional*.

Definition. *Let X be a real vector space (not necessarily normed). A map $p : X \to \mathbf{R}$ is called a* **sublinear functional** *if it satisfies the following two properties:*

$$p(x+y) \le p(x) + p(y) \quad \forall x, y \in X \quad \text{(subadditive)}$$
$$p(\alpha x) = \alpha p(x) \quad \forall \alpha \ge 0,\ \forall x \in X \quad \text{(positive homogeneity)}$$

Note that "norm" is a sublinear functional.

The next result is a useful property of sublinear functionals.

Lemma. *Let $M \subset X$ be a subspace, and f a linear functional on M, such that for some sublinear functional p,*

$$f(x) \le p(x) \quad \forall x \in M.$$

Let x_o be a fixed element of X. Then, for any real number c, the following are equivalent :

$$f(x) + \lambda c \le p(x + \lambda x_o) \quad \forall x \in M, \lambda \in \mathbf{R} \tag{1}$$

$$-p(-x - x_o) - f(x) \le c \le p(x + x_o) - f(x) \quad \forall x \in M \tag{2}$$

Furthermore, there is a real number c satisfying (2) and hence (1).

Proof : To show that (1) $\Rightarrow$ (2), first set $\lambda = 1$, and then set $\lambda = -1$ and replace x by $-x$; then (2) is immediate. To show the converse, first take $\lambda > 0$, and replace x by x/λ on the RHS inequality, to obtain (1). Now take $\lambda < 0$, and replace x by x/λ on the LHS inequality, to arrive again at (1). For $\lambda = 0$, (1) is always satisfied, from the definitions of f and p. To produce the desired c, let x, y be arbitrary elements out of M. Then,

$$f(x) - f(y) = f(x-y) \le p(x-y)$$
$$\le p(x + x_o) + p(-y - x_o) \quad \text{by subadditivity}$$
$$\Rightarrow \qquad -p(-y - x_o) - f(y) \le p(x + x_o) - f(x) \quad \forall x, y \in M$$
$$\Rightarrow \qquad \sup_{y \in M} \{-p(-y - x_o) - f(y)\} \le \inf_{x \in M} \{p(x + x_o) - f(x)\}$$

Any c between the "sup" and "inf" values will do it. ⋄

Now we have the main theorem.

Theorem (Hahn-Banach). *Given a real linear vector space X and a subspace M, let p be a sublinear functional on X, and f a linear functional on M such that $f(x) \le p(x) \quad \forall x \in M$. Then, there is a linear functional F on X such that $F = f$ on M and $F \le p$ on all of X.*

Proof. Let $x_0 \notin M, x_0 \in X$, and consider the subspace M_0 consisting of all elements $x + \lambda x_0, x \in M, \lambda \in \mathbf{R}$. We may extend f to a linear functional on M_0 by defining

$$f_0(x + \lambda x_0) = f(x) + \lambda c$$

where c is any real number. Now choose c such that it satisfies condition (2) in the previous Lemma. Then,

$$f_0(x + \lambda x_0) \leq p(x + \lambda x_0), \quad \Longleftrightarrow \quad f_0 \leq p \quad \text{on } M_0$$

Hence, we have seen how to extend f from M to M_0, which is of dimension one higher than that of M (assuming that M is finite-dimensional).

1. If X is normed and separable, then we can find a countable set of vectors $\{x_0, x_1, \ldots\}$, all linearly independent and **not** in M, so that every element in X can be approximated to any degree by linear combinations of vectors out of M and $\{x_0, x_1, \ldots\}$. Let $M_i = M_{i-1} + \lambda x_i$, which is a subspace. If $y \in M_i$, we can write it as $y = x + \lambda x_i$, where $x \in M_{i-1}$ and $\lambda \in \mathbf{R}$. Define

$$f_i(x + \lambda x_i) = f_{i-1}(x) + \lambda c_i$$

where c_i is as in the Lemma with f replaced by f_{i-1}. This then shows how we can extend f recursively to a countable dense subset of X. Call this extension g, which is naturally linear. By construction, $g \leq p \quad \forall x \in S$, where S is a dense subset of X. Now, to extend g to X, let $x \in X$ be given and $\{s_n\}$ be a sequence out of S converging to x. Then, the limit $\lim_{n\to\infty} g(s_n)$ is well defined; call this $F(x)$. This defines F pointwise. F is obviously linear. Furthermore, since $g(s_n) \leq p(s_n) \quad \forall s_n \in S$, we have

$$F(x) \leq p(x) \quad \forall x \in X \ .$$

2. If X is not a separable normed space, we cannot find an ordered set of vectors to form a basis for a dense subset. Then, the proof will have to be modified, where the strict ordering is replaced by partial ordering, and *Zorn's Lemma* is used. *Zorn's Lemma* says that: "If P is a partially ordered set in which every chain has an upper bound, then P possesses a *maximal* element." This is equivalent to the *Axiom of Choice*, which says the following: "Given a nonempty class of disjoint nonempty sets, a set can be formed which contains precisely one element taken from each set in the given class." To use this in our proof, let $\mathcal{C}$ be the collection of all pairs (h, H) where h is an extension of f to the subspace $H \supset M$, and $h \leq p$ on H. Partially order $\mathcal{C}$ by $(h_1, H_1) \leq (h_2, H_2)$ iff $H_1 \subset H_2$ and $h_1 = h_2$ on H_1. Then, every chain in $\mathcal{C}$ has an upper bound (consider the union of all subspaces in the chain). This implies by *Zorn's Lemma* that $\mathcal{C}$ has a maximal element. Call this $(F, \mathcal{F})$. If $\mathcal{F} \neq X$, then this means that we can extend F to a larger subspace – but this contradicts its maximality. Hence F is defined on X. $\diamond$

The two corollaries below follow from the *Hahn-Banach* theorem. I provide a proof for only the first one.

Corollary 1. *Let f be a continuous linear functional defined on a subspace M of X, a normed linear space. Then, there exists $F \in X^*$ such that $\|f\|_M = \|F\|_X$ and F is an extension of f.*

Proof. Take $p(x) = \|f\|_M \, \|x\|_X$ in the Hahn-Banach Theorem, which is clearly a sublinear functional. Now note that

$$F(x) \leq p(x) = \|f\|_M \, \|x\|_X \quad \text{and} \quad -F(x) = F(-x) \leq \|f\|_M \, \|x\|_X \quad \forall\, x \in X \,,$$

which leads to

$$|F(x)| \leq \|f\|_M \, \|x\|_X \quad \Rightarrow \quad \sup_{x\in X, \|x\|\leq 1} |F(x)| \leq \|f\|_M \,.$$

But since $\sup_{x\in X, \|x\|\leq 1} |F(x)| \geq \sup_{x\in M, \|x\|\leq 1} |F(x)| = \|f\|_M$, we actually have equality above. Since F is bounded, it is also continuous. $\diamond$

Corollary 2. *Let $x_o \in X$, a normed linear space. Then, there exists a nonzero bounded linear functional F on X such that*

$$F(x_o) = \|F\| \|x_o\| \ .$$

The converse of this last result does not generally hold; that is, given $F \in X^*$, we may not be able to find $x \in X$ such that $F(x) = \|F\| \|x\|$. See, for example, Example 1 on page 113 of the text by Luenberger. We will see (in class) that for some (but not all) Banach spaces this converse indeed holds.

$\diamond \quad \diamond \quad \diamond$