ASSIGNMENT 5

Reading Assignment: Text: Chapter 5. Correspondence 12

Notice : This is the last homework assignment before the midterm exam, which is scheduled for Friday, March 15 (3:30 pm - 5:00 pm), in 141 CSL.

Problems (to be handed in): Due Date: **Tuesday, March 12**.

39. Let H be the Hilbert space of Lebesgue-measurable functions on $[0,\infty)$, endowed with the inner product

$$(x,y) = \int_0^\infty e^{-4t}\, x(t)y(t)\, dt\,, \quad x,y \in H\,.$$

Define a functional f on H by

$$f(x) = \int_0^\infty e^{-4t}\, dt \int_0^t K(t,s)\, x(s)\, ds$$

where $K(t,s)$ is a square-integrable (so-called *Hilbert-Schmidt*) kernel on $[0,\infty) \times [0,\infty)$, that is

$$\int_0^\infty \int_0^\infty |K(t,s)|^2\, dt\, ds < \infty\,.$$

i) **Show** that f is linear and continuous.

ii) **Determine** an element $y \in H$ such that $f(x) = (x,y)$. Is the solution unique?

40. Define a functional f on $L_4[0,3]$ by

$$f(x) = \int_0^3 dt \int_0^t K(t,s)\, x(s)\, ds$$

i) **Obtain** tight conditions on the kernel $K(\cdot,\cdot)$ such that f is linear and continuous.

ii) Assuming that K satisfies the condition(s) in (i) above, **determine** $\|f\|$, the norm of f (in terms of K).

iii) Now **compute** $\|f\|$ when $K(t,s) = t - s$.
Note : $L_4[0,3]$ is **not** a Hilbert space.

41. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and $L_2(\Omega, \mathcal{P}; L_2[0,2])$ be the Hilbert space of second-order random processes defined on $[0,2] \times \Omega$. Consider the mappings:

$$g : L_2(\Omega, \mathcal{P}; L_2[0,2]) \to L_2(\Omega, \mathcal{P}; \mathbf{R}); \quad f : L_2(\Omega, \mathcal{P}; \mathbf{R}) \to \mathbf{R}$$

defined by

$$g(x) = \int_0^2 b(t)\, x(t;\omega)\, dt\,, \quad b \in L_2[0,2]\,; \qquad f(y) = E[y]\,.$$

Let h be the composite mapping $f \circ g : L_2(\Omega, \mathcal{P}; L_2[0,2]) \to \mathbf{R}$.

i) Show that both f and h are bounded linear functionals.

ii) If $(\cdot,\cdot)$ and $<\cdot,\cdot>$ denote the natural inner products on $L_2(\Omega, \mathcal{P}; \mathbf{R})$ and $L_2(\Omega, \mathcal{P}; L_2[0,2])$, respectively, **find** elements $\bar{y}$ and $\bar{x}$ in these respective spaces so that

$$f(y) = (y, \bar{y})\,, \quad h(x) = < x, \bar{x} > \,.$$

42. Let $X = C[0,1]$ with the standard maximum norm, and $f \in X^*$ be defined by

$$f(x) = 2x(1/3) + 3x(1/2) + 4\int_0^1 t x(t)\, dt$$

i) Compute $\|f\|$, the norm of f.

ii) Obtain a function of bounded variation on $[0,1]$, say $v(\cdot)$, such that $v(0) = 0$ and

$$f(x) = \int_0^1 x(t)\, dv(t)$$

43. Let $\mathbf{c_0}$ be the space of all infinite sequences of real numbers converging to *zero*, endowed with the *max* norm. Let $x = \{\xi_i\}$ be a fixed element of $\mathbf{c_0}$, given by

$$\xi_i = \begin{cases} -1 & i = 1 \\ 2/i & i \geq 2 \end{cases}$$

i) Find a linear functional, f, on $\mathbf{c_0}$, of unit norm, such that $f(x) = \|x\|$. Is the solution **unique**?

(Note: A linear functional f is said to be **aligned** with x if $f(x) = \|f\| \cdot \|x\|$. Hence, here we are looking for an f with unit norm, which is aligned with x.)

ii) Find an element g of $\mathbf{c_0^*}$, the normed dual of $\mathbf{c_0}$, such that g has unit norm and $g(x) = 0$. Is the solution **unique**?

(Note: The property $g(x) = 0$ is known as the linear functional g being **orthogonal** to x.)

44. Let x be a continuous function on $[0,3]$, given by

$$x(t) = \begin{cases} 2 & 0 \leq t \leq 1/2 \\ 4 - 4t & 1/2 \leq t \leq 3/2 \\ 4t - 8 & 3/2 \leq t \leq 5/2 \\ 2 & 5/2 \leq t \leq 3 \end{cases}$$

Letting $X = C[0,3]$, find two linear functionals, $x^* \in X^*$ and $y^* \in X^*$, both of unit norm, such that x^* is **aligned** with x, and y^* is **orthogonal** to x. Are the solutions **unique** ?

45. Let $g_1, g_2, \ldots, g_n$ be linearly independent linear functionals on a vector space X. Let f be another linear functional on X such that for every $x \in X$ satisfying $g_i(x) = 0$, $i = 1, 2, \ldots, n$, we have $f(x) = 0$. **Show** that there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$f = \sum_{i=1}^{n} \lambda_i \, g_i \, .$$

Hint : Use the Hahn-Banach Theorem (extension form; Correspondence 12).

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