ASSIGNMENT 4

Reading Assignment: Text: Chapter 4. Correspondence 10

Suggested Reading: Curtain & Pritchard: Chp 5 (pp. 75-84).
Review probability theory and stochastic processes from any (graduate) text of your choice.

Notice : **On February 28, we will start class at 9:30 am**

Problems (to be handed in): Due Date: **Thursday, February 28**.

This first problem of this set is related to the topic of "wavelets" (which I briefly introduced in class), but no prior knowledge of wavelets is necessary to solve it.

33. Let J be an index set, and $\{\xi_j\}_{j\in J}$ a family of functions in a (complex) Hilbert space H. This family is called a *frame* if there exist constants $A > 0$, $B < \infty$ such that for all $f \in H$,

$$A\,\|f\|^2 \le \sum_{j\in J} |(f,\xi_j)|^2 \le B\,\|f\|^2$$

Here A and B are called *frame bounds*. If $A = B$, then the frame is said to be a *tight frame*. (Note that the family $\{\xi_j\}_{j\in J}$ is not necessarily orthogonal, or even linearly independent.)
i) Show that if the family $\{\xi_j\}_{j\in J}$ constitutes a tight frame, then

$$f = A^{-1} \sum_{j\in J} (f,\xi_j)\,\xi_j$$

Hint: First verify the following identity in H, which will prove useful in establishing the desired result: For any $f, g \in H$:

$$4(f,g) = \|f+g\|^2 - \|f-g\|^2 + \imath\|f+\imath g\|^2 - \imath\|f-\imath g\|^2$$

ii) To show that it is possible for $\{\xi_j\}_{j\in J}$ to be a tight frame, without being orthogonal or linearly independent, consider the following (counter-)example:

$$H = \mathbf{C}^2, \quad \xi_1 = (0,1)^T, \quad \xi_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T, \;\; \xi_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T$$

Show that the triplet $\{\xi_1, \xi_2, \xi_3\}$ does indeed constitute a tight frame.
What is the frame bound A?

iii) Prove that, again for the general case, if $\{\xi_j\}_{j\in J}$ is a tight frame, with frame bound $A=1$, and if $\|\xi_j\|=1\ \forall j\in J$, then the ξ_j's constitute an orthonormal basis for H.

The remaining problems in this set are all on the topic of Hilbert Spaces of Random Variables and Stochastic Processes.

34. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space, and $L_2(\Omega,\mathcal{P};\mathbf{R}^n)$ be the Hilbert space of second-order random vectors (of dimension n) defined on $(\Omega,\mathcal{F},\mathcal{P})$, with inner product

$$(x,z)=E[x^TQz]$$

where Q is a given (fixed) positive-definite matrix of dimension $n\times n$. Let $\{y_0,\ldots,y_i\}$ be $m-$dimensional random vectors defined on $(\Omega,\mathcal{F},\mathcal{P})$, which are uncorrelated and have zero mean. Let $\mathcal{M}_{nm}$ be the class of all $n\times m$ matrices with bounded entries, and consider the following optimization problem for a given $x\in L_2(\Omega,\mathcal{P};\mathbf{R}^n)$:

$$\|x-\sum_{j=0}^{i}\hat{K}_jy_j\|=\inf_{K_j\in\mathcal{M}_{nm}}\|x-\sum_{j=0}^{i}K_jy_j\|.$$

i) Solve for the optimal $\hat{K}_j,\ j=0,\ldots,i$. Is the solution unique?

ii) Let $\epsilon_k=\|x-\sum_{j=0}^{k}\hat{K}_jy_j\|^2$, and obtain a recursive (linear first-order difference) equation for ϵ_k.

35. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space, and x,y_1,y_2 three zero-mean second-order random variables defined on this space, with y_1 and y_2 uncorrelated. Let $\mathcal{Z}$ be the class of random variables $z=a_1y_1+a_2y_2$, where the coefficients a_1 and a_2 are restricted to be nonnegative (that is, $a_1\geq 0, a_2\geq 0$). We seek a best approximation to x in $\mathcal{Z}$ in the minimum mean square sense, that is an $\hat{x}\in\mathcal{Z}$ such that

$$\inf_{z\in\mathcal{Z}}E\left[(x-z)^2\right]=E\left[(x-\hat{x})^2\right]$$

i) Formulate this problem as one of minimum distance to a convex set in a Hilbert space.

ii) Does there exist a unique solution? Justify your answer.

iii) Compute $\hat{x}$ and $E\left[(x-\hat{x})^2\right]$ when

$$E[(y_1)^2]=E[(y_2)^2]=E[x^2]=1,\quad E[y_1x]=0.2,\ E[y_2x]=-0.5\,.$$

36. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space, and y a random variable on it, with $E[y]=1$ and $E[y^2]=2$. We wish to find another random variable, x, on the same probability space, with **minimum second moment**, and satisfying the constraints
$E[xy]=2$ and $E[x]=-1$.

i) Does this problem admit a solution? Is it unique? Justify your answers.

ii) Obtain the solution if it exists.

iii) What would the solution be if the second equality constraint is replaced by the inequality constraint: $E[x] \geq -1$

37. Let Y_1 and Y_2 be uncorrelated second-order random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $L_2(\Omega, \mathcal{F}; C[0,1])$ be the space of all parametrized (in t) random variables (equivalently, stochastic processes) $X(t;\omega)$, where for fixed $t \in [0,1]$, $X(t;\cdot)$ is a second-order random variable on $(\Omega, \mathcal{F}, \mathcal{P})$ and for fixed $\omega \in \Omega$, $X(\cdot;\omega) \in C[0,1]$. Define the inner product on $L_2(\Omega, \mathcal{F}; C[0,1])$ by

$$(X, Z) = E\big[\int_0^1 X(t;\omega)\, Z(t;\omega)\, w(t)\, dt\big];$$

where $w(\cdot) > 0$ is in $C[0,1]$. Determine a stochastic process $\widehat{X}(t;\omega) \in L_2(\Omega, \mathcal{F}; C[0,1])$ which has **minimum norm** and satisfies the equalities:

$$E\big[\int_0^1 \widehat{X}(t;\omega)\, k_i\,(t) Y_i(\omega)\, dt\big] = c_i\,, \quad i = 1, 2\,,$$

where k_1, k_2 are linearly independent elements out of $C[0,1]$, and c_1, c_2 are given constants.

38. Let X be a second-order random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $Y(t;\omega)$ be a second-order stochastic process defined on the same probability space, with $t \in [0,2]$, which is correlated with X, with the cross-correlation function given by $R_{XY}(t) = E[XY(t)]$. Further let $R_{YY}(t,s)$ denote the auto-correlation function of Y. We are interested in finding a linear least squares (*l.l.s.*) estimate of X given the measurement process $Y(t;\omega)$ over the interval $[0,2]$, that is an estimate in the form

$$m(\omega) = \int_0^2 K(t) Y(t;\omega)\, dt$$

for some function $K(\cdot)$.

i) **Show** that there exists a unique such *l.l.s.* estimate, and obtain the equation satisfied by a corresponding optimum $K(\cdot)$ in terms of R_{XY} and R_{YY}. Under what conditions is the optimum $K(\cdot)$ **unique**?

ii) **Redo** (i) above when $K(\cdot)$ is restricted to be a constant (independent of time).

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