

ASSIGNMENT 3**Reading Assignment:**

Text: Chp 3; Correspondence # 6.

Recommended Reading:

Curtain & Pritchard: Chp 4 (pp. 55-64), Chp 5 (pp. 75-84).

Balakrishnan: Chapters 1 and 2.

Liusternik & Sobolev: Chapter 2 (pp. 73-83).

Advance Reading:

Text: Chapter 4; Review probability theory and stochastic processes from any (graduate) text of your choice.

Problems (to be handed in): Due Date: **Tuesday, February 19.**

The first two problems below are on games, and application of fixed point theorems in that context.

23. Consider the zero-sum game with objective function

$$F : [0, 1] \times [0, 1] \rightarrow \mathbf{R} : \quad F(x, y) = -2x^2 + y^2 + 3xy - x - 2y$$

which is to be maximized with respect to $x \in [0, 1]$ and minimized with respect to $y \in [0, 1]$.

(a) Is F convex-concave? Does the game admit a saddle point? Justify your answers.

(b) Obtain the saddle-point solution (if there exists one). Is it unique?

(c) Consider the sequence $\{x_k, y_k\}$ generated by:

$$F(x_{k+1}, y_k) = \max_{x \in [0, 1]} F(x, y_k); \quad F(x_{k+1}, y_{k+1}) = \min_{y \in [0, 1]} F(x_{k+1}, y); \quad k = 0, 1, \dots$$

where $y_0 \in [0, 1]$ is arbitrary. Does this sequence converge, and if it does is the limit the saddle-point solution (if one exists)?

24. Fan's fixed-point theorem is also useful in the proof of existence of a noncooperative equilibrium (*Nash equilibrium*) in *nonzero-sum games*. Nonzero-sum games could in fact involve more than two players, but here let us assume that there are actually two players. These are games where the players have different objective functionals, which do not add up to zero as in zero-sum games—thus the name *nonzero-sum*).

Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, and $K_X \subset X$ and $K_Y \subset Y$ be nonempty convex compact subsets of X and Y , respectively. Player 1 seeks to minimize $F_1(x, y)$ with respect to $x \in K_X$, and Player 2 seeks to minimize $F_2(x, y)$ with respect to $y \in K_Y$. We say that a pair $(x^* \in K_X, y^* \in K_Y)$ constitutes a *Nash equilibrium*, if :

$$\min_{x \in K_X} F_1(x, y^*) = F_1(x^*, y^*) \quad \text{and} \quad \min_{y \in K_Y} F_2(x^*, y) = F_2(x^*, y^*)$$

(a) Prove that a two-player nonzero-sum game as formulated above admits a Nash equilibrium if

i) F_1 and F_2 are both continuous on $X \times Y$; and

ii) F_1 is convex in $x \in K_X$ for each $y \in K_Y$, F_2 is convex in $y \in Y$ for each $x \in X$.

(b) Obtain the set of all Nash equilibria for the nonzero-sum game where

$$X = Y = \mathbb{R}, K_X = K_Y = [0, 1]; \quad F_1(x, y) = (y + 1)x^2 - 2x, \quad F_2(x, y) = 2xy - y$$

The next seven problems involve inner products, Hilbert spaces and optimization in such spaces.

25. Let \mathbf{H} be the space of all $m \times m$ matrices with complex-valued entries, with addition and multiplication defined as the standard corresponding operations with matrices, and with a candidate inner product of two matrices A, B defined as

$$(A, B) = \text{Trace}(A^T Q \bar{B})$$

where A^T denotes the transpose of the matrix A ; \bar{B} denotes the complex conjugate of B ; and $Q \in \mathbf{H}$ is a Hermitian positive-definite matrix, that is $Q^T = \bar{Q}$ and Q has only positive real eigenvalues. Prove that (A, B) as defined above is an inner product on \mathbf{H} .

26. (a) Let X denote the space of continuous functions defined on the closed finite interval $[1, 4]$ of the real line. Let (\cdot, \cdot) be a mapping from $X \times X$ onto \mathbb{R} , defined by

$$(x, y) = \int_1^4 \int_1^4 s^2 x(s) t^2 y(t) ds dt$$

Determine whether (\cdot, \cdot) is an inner product on X . Justify your answer.

(b) Now study whether (or not) the following is an inner product on X (where X is the same space as above). Again justify your answer.

$$(x, y) = \int_1^4 t^3 x(t) y(t) dt$$

27. We seek an affine function $m(t) = a + bt$ that minimizes the integral

$$F(m) = \int_{-1}^2 [t^3 - m(t)]^2 dt.$$

(a) Formulate this as a projection problem in a Hilbert space H , by clearly identifying both H and the subspace M .

(b) Obtain the solution. Is it unique?

(c) What is the minimum value of F ?

28. Repeat *Problem 27* above with F replaced by

$$F(m) = \int_{-1}^2 t^2 [t^3 - m(t)]^2 dt.$$

29. Obtain the solution to the problem of minimizing the functional $\int_1^2 t x^2(t) dt$ subject to two constraints:

$$\int_1^2 x(t) t^{1/3} dt = 1 \quad \text{and} \quad \int_1^2 x(t) t^{2/3} dt = -1$$

Is the solution unique?

30. Using the *Projection Theorem*, solve the finite-dimensional optimization problem:

$$\text{minimize } x^T Q x \quad \text{subject to } Ax = b$$

where x is an n -vector, Q a positive-definite (symmetric) matrix, A an $m \times n$ matrix (with $m < n$), and b an m -vector.

[You should be able to obtain a closed-form expression in terms of A , Q and b .]

31. Given a function $x \in L_2[a, b]$, where $[a, b]$ is a finite interval, we seek a polynomial p of degree n or less which minimizes the functional

$$\int_a^b |x(t) - p(t)|^2 dt$$

subject to the constraint

$$\int_a^b p(t) dt = 0.$$

(a) Show that this problem admits a unique solution.

(b) Show that the problem can be solved by first finding the polynomial q of degree n or less, say q° , which minimizes

$$\int_a^b |x(t) - q(t)|^2 dt$$

without any constraint, and then finding the polynomial p of degree n or less which minimizes the functional

$$\int_a^b |q^\circ(t) - p(t)|^2 dt$$

subject to the original constraint

$$\int_a^b p(t) dt = 0.$$

The next (and last) problem is related to the topic of minimum distance to a convex set, which is discussed in section 3.12 of the Text; see in particular Theorem 1 on page 69.

- 32.** Consider the problem of finding the vector x of minimum norm satisfying the inequality constraints:

$$(x, y_i) \geq c_i, \quad i = 1, 2, \dots, n$$

where the y_i 's are linearly independent, and the c_i 's are given constants.

i) Show that this optimization problem admits a unique solution.

ii) Show that a necessary and sufficient condition for a vector $x = \sum_{i=1}^n a_i y_i$ to constitute a solution to this problem is that the vector a with components a_i satisfy $G^T a \geq c$; $a \geq \theta$, and that $a_i = 0$ if $(x, y_i) > c_i$. Here G is the Gram matrix of $\{y_1, y_2, \dots, y_n\}$.

◇ ◇ ◇