ECE 580 **SPRING 2019**

Correspondence # 4 January 28, 2019

ASSIGNMENT 2

Reading Assignment: Text: Chp 10 (pp. 272-277); Correspondence # 3.

Recommended Reading: Curtain & Pritchard: Chapters 1 (pp. 20-22), 4 (pp. 55-64); Balakrishnan: Chapter 1, and Chapter 2 (pp. 54-57). Liusternik & Sobolev: Chapter 1 (pp. 26-44).

Advance Reading: Text: Chapter 3

Problems (to be handed in): Due Date: **Tuesday, February 5**

11. Consider the space ℓ_p' consisting of all ordered numbers $(\xi_1, \xi_2, \ldots, \xi_{k_1})$ where k_1 is a natural number and ξ_i's are arbitrary real numbers. If

$$x := (\xi_1, \xi_2, \ldots, \xi_{k_1}) \quad y := (\nu_1, \nu_2, \ldots, \nu_{k_2}) \ , \quad k_2 \geq k_1,$$

we introduce a metric by

$$\rho(x,y) = (\sum_{i=1}^{k_1} |\xi_i - \nu_i|^p + \sum_{j=k_1+1}^{k_2} |\nu_j|^p)^{1/p}, \quad 1 \leq p < \infty.$$

Show that ℓ_p' is **not** a complete metric space.
Hint: You have to show that there exists a Cauchy sequence in ℓ_p', whose limit is not in ℓ_p'; consider, for instance, the sequence:

$$x_1 = \{1\},\ x_2 = \{1, \frac{1}{2}\},\ x_3 = \{1, \frac{1}{2}, \frac{1}{2^2}\}, \ldots,\ x_n = \{1, \frac{1}{2}, \ldots, \frac{1}{2^{n-1}}\}, \ldots$$

12. Let $f : C[0,2] \to \mathbf{R}$ be a functional defined by

$$f(x) = \max_{0 \leq t \leq 2} x(t)$$

(a) **Show** that f is **continuous**.
(b) Is f **uniformly** continuous?
Note: A transformation $T : X \to Y$, where both X and Y are normed linear spaces, is **uniformly** continuous, if in the (ϵ, δ) definition of continuity, δ depends on only ϵ, and not on $x \in X$.

13. This problem is similar to the preceding one with a different interval and a different function f. Let $f : C[0, 4] \to \mathbf{R}$ be a functional defined by

$$f(x) = \max_{0 \le t \le 4} \big(x(t)\big)^3$$

(a) **Show** that f is **continuous**.

(b) Is f **uniformly** continuous?

14. As to be discussed in class on Tuesday (January 29), the Banach space $L_p[a, b]$ is **separable**, for $1 \le p < \infty$ and for any finite interval $[a, b]$ (see also Example 4, on page 43 of the text). This result does not hold, however, if the interval is infinite, that is $(-\infty, \infty)$, and the norm adopted is

$$\|x\| = \left(\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^p \, dt \right)^{\frac{1}{p}}$$

We denote the space in this case by $L_p(-\infty, \infty)$.

Construct a counterexample to show that $L_2(-\infty, \infty)$ is not separable.

Hint : Try the continuum of elements $\{\sin \alpha t,\ \alpha \in (-\infty, \infty)\}$ as a candidate to generate a counterexample.

15. Let $(X, \| \cdot \|)$ be a normed linear space, and $x_1, x_2, \ldots, x_n$ be linearly independent vectors from X. For fixed $y \in X$, show that there exist coefficients $a_1, a_2, \ldots, a_n$ minimizing the quantity

$$\|y - a_1 x_1 - a_2 x_2 \cdots - a_n x_n\|.$$

Hint : First show that the search for the minimizing coefficients can be restricted to a closed and bounded finite-dimensional set, and then use the Weierstrass theorem (p. 40 of the text).

[This is Problem 9 on page 44 of the text.]

16. In *Problem 15* above, let $X = L_2[0, 1]$, $n = 2$, $x_1(t) = 1, x_2(t) = t, y(t) = t^3, 0 \le t \le 1$. Obtain the optimum values of a_1 and a_2.

17. Repeat *Problem 16* for the case when $X = C[0, 1]$ (with the Chebyshev (maximum) norm). [*Note: This requires a much more involved analysis than the previous one.*]

The next five problems are on fixed points (topic of Correspondence # 3).

18. Consider the fixed-point (FP) equation

$$x(t) = \frac{1}{2} t^3 + \alpha \sin \pi x(t)$$

defined over the interval $t \in [-2, 2]$, with $x \in C[-2, 2]$, and α a positive constant (a parameter). For what values of α does there exist a *unique* continuous function $x(\cdot)$ on $[-2, 2]$ which solves the FP equation. Show (prove) that for these values of α a solution indeed exists and is unique.

Hint : Use a contraction mapping type argument, applied to a subset of $C[-2, 2]$, which comprises all uniformly bounded functions, such as functions satisfying the bound $|x(t)| \leq \beta$, for some β.

19. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a transformation defined by $T(x) = Ax + b\,,$ where A is an $n \times n$ matrix with real entries a_{ij}, and b is a given vector in $\mathbf{R}^n$, with components b_i's.

(a) Under what conditions on the a_{ij}'s and b_i's is T a contraction when the norm on $\mathbf{R}^n$ is the Euclidean one, that is $\|x\| = \left\{ \sum_{i=1}^n (x_i)^2 \right\}^{1/2}$?

(b) Under what conditions on the a_{ij}'s and b_i's is T a contraction when the norm on $\mathbf{R}^n$ is the maximum norm, that is $\|x\| = \max_i |x_i|$? Are these conditions more or less restrictive than the ones obtained in part (a)?

(c) We now wish to compute a *fixed point* of T by using the iteration (successive approximation)

$$x_{(i+1)} = Ax_{(i)} + b, \quad i = 0, 1, \ldots$$

where $x_{(0)}$ is an arbitrary starting point. Based on the results you obtained in parts (a) and (b) above, what can you deduce as the conditions on the a_{ij}'s and b_i's for this sequence to converge to a fixed point of T.

(d) Show that the condition you obtained in part (c) above holds for the special 3-dimensional case when

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.2 \\ 0.6 & 0.2 & 0.1 \\ 0.2 & 0.7 & 0 \end{pmatrix} \; ; \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and obtain the fixed point of T numerically (to 5 significant places) by starting the iteration at **(i)** the origin (i.e., $x_{(0)} = \theta$), and **(ii)** $x_{(0)} = (\,-1 \quad 1 \quad -1\,)'$.

20. Consider the following integral equation where λ is a constant parameter:

$$x(t) + \lambda \int_0^2 (t - s)\, x(s)\, ds = t^3\,, \qquad 0 \leq t \leq 2$$

(a) If the integral equation is defined on $L_2[0, 2]$, for what values of λ does it admit a unique solution?

(b) Is the solution continuous, i.e., does it belong to $C[0, 2]$?

(c) If the integral equation is instead defined on $L_1[0,2]$, for what values of λ does it admit a unique solution using Banach's contraction mapping theorem? Would this solution be different than the one in (a)?

21. Consider the following linear equation defined on ℓ_1:

$$\lambda \sum_{m=1}^{\infty} a_{nm}\, x_m = 3^{-2n} + x_n\,, \quad n = 1, 2, \ldots$$

where λ is again a parameter, and a_{nm}'s are real numbers.

(a) Under what conditions on λ and the a_{nm}'s does this equation admit a unique solution $x^\circ \in \ell_1$? [Use the contraction mapping theorem to solve this problem.]

(b) Now let

$$a_{nm} = \begin{cases} \dfrac{1}{2n^2} & m = n \\ \dfrac{1}{4m^2 n^2} & m \neq n \end{cases}$$

Using the result of part (a) above, obtain the conditions on λ such that the map $T : \ell_1 \to \ell_1$, defined by

$$T(x)_n := \lambda \sum_{m=1}^{\infty} a_{nm}\, x_m - 3^{-2n}$$

is **(i)** contraction, **(ii)** nonexpansive.

22. Let X be a Banach space with norm $\|\cdot\|$, and let S be a compact subset of X. Let T be a contractive (*not contraction*) mapping of S into itself. Show that T has a unique fixed point, and it can be obtained using the iteration

$$x_{(n+1)} = T(x_{(n)})\,, \quad n = 0, 1, \ldots$$

with an arbitrary starting point $x_{(0)} \in S$.

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