ECE 580 SPRING 2019
Correspondence # 1 January 17, 2019

ASSIGNMENT 1

Reading Assignment: Text: Chapters 1 & 2. [This includes advance reading.]

Recommended Reading: Curtain & Pritchard: Chapter 1.

Problems (to be handed in): Due Date: **Thursday, January 24**.

1. Let M and N be subspaces in a vector space. Show that $[M \cup N] = M + N$, where the set operations "$[\cdot]$" and "$+$" are as defined in the text (also introduced in class). [This is Problem 3 on page 43 of the text.]

2. A *convex combination* of the vectors $x_1, x_2, \ldots, x_n$ is a linear combination of the form $\alpha x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ where $\alpha_i \geq 0$ for each i, and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Given a set S in a vector space, let K be the set of vectors consisting of all convex combinations from S. Show that $K = \mathrm{co(S)}$, where $\mathrm{co}(\cdot)$ is defined on p. 18 of the text.
[This is Problem 4 on page 43 of the text.]

3. I introduced in class today the Hölder inequality for sequences. Its counterpart for continuous (or more generally integrable) functions is given in Theorem 3 of the text (pp. 32-33). I also mentioned in class that the Hölder inequality is a versatile tool in some classes of optimization problems. This problem, as well as the next two, constitute illustrations of this powerful tool.
Obtain a continuous function $x(\cdot)$ on the interval $[-1, 1]$ which **maximizes** the integral

$$\int_{-1}^{1} t^3 x(t)\, dt$$

subject to the constraint

$$\int_{-1}^{1} |x(t)|^3 dt \leq 2\,.$$

Hint: Use Hölder's inequality applied to a specific normed linear space.

4. Consider the same problem as in *Problem 3* above, but now we want to **minimize** the integral

$$\int_{-1}^{1} t^3 x(t)\, dt$$

subject to the same inequality constraint. What is the optimal (minimizing) solution?

5. Find a function $x \in L_2[0,3]$ of unit norm, i.e.,

$$\int_0^3 |x(t)|^2 \, dt = 1 \,,$$

which **minimizes** the functional

$$f(x) = \int_0^3 x(t) \cos \pi t \, dt \,.$$

Is the solution **unique**?

6. A normed vector space is said to be *strongly normed* (equivalently, *strictly normed*) if the equality $\|x+y\| = \|x\| + \|y\|$ implies that either $y = \theta$ or $x = \alpha y$ for some scalar α. Show that $L_p[0,1]$ is strongly normed, for $1 < p < \infty$.
[This is part of Problem 10 on page 44 of the text.]

7. Let $(X, \|\cdot\|)$ be a normed linear space, and $x_1, x_2, \ldots, x_n$ be linearly independent vectors from X. For fixed $y \in X$, we are interested in minimizing the quantity

$$\|y - a_1 x_1 - a_2 x_2 \cdots - a_n x_n\|$$

with respect to the real parameters $a_1, a_2, \ldots, a_n$. Note that this is a problem of approximation of $y \in X$ in the span of $x_1, x_2, \ldots, x_n$. Of course, if y is in that subspace, then we have the trivial case with the minimum value being zero. Assume therefore that this is not the case, and further assume that there is a solution to this optimization problem (one can actually show (using the Weierstrass theorem (p. 40 of the text)) that this is in fact always the case, but showing this is not part of this problem–it will be assigned as part of the second problem set). Now, show that if X is strongly normed (see *Problem 6* above), then the solution here is **unique**.
Hint: Use an argument of contradiction—assuming that there are two solutions, and showing that the strong-norm property leads to a contradiction.

8. Suppose that $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \in \ell_{p_o}$ for some p_o, $1 \leq p_o < \infty$.
(a) Show that $x \in \ell_p$ for all $p \geq p_o$.
(b) Show that $\|x\|_\infty := \sup_i |\xi_i| = \lim_{p \to \infty} \|x\|_p$.

(For terminology and notation, see section 2.10 of the text; part (b) was also discussed (but not proven) in class today.) [This is Problem 12 on page 44 of the text.]

9. Given a linear vector space X, a function $\rho(\cdot,\cdot) : X \times X \to I\!R$ is a **metric** on X if it satisfies the following three properties: (i) $\rho(x,y) \geq 0 \; \forall x, y \in X$, and is equal to 0 iff

$x = y$; (ii) $\rho(x,y) = \rho(y,x)\ \forall x, y \in X$; (iii) $\rho(x,z) \le \rho(x,y) + \rho(y,z)\ \forall x, y, z \in X$. This makes X a metric space, denoted by (X, ρ).
Now, given a normed vector space $(X, \|\cdot\|)$, let us introduce a function $n : X \times X \to \mathbb{R}$ by

$$n(x,y) = \frac{\|x-y\|}{1+\|x-y\|}\ .$$

Show that $n(\cdot,\cdot)$ provides a metric on X.

10. A real-valued functional $g(x)$ defined on a real vector space X is called a *seminorm* if:
(i) $g(x) \ge 0$ for all $x \in X$;
(ii) $g(\alpha x) = |\alpha| \cdot g(x)$ for all $x \in X$, and all real numbers α;
(iii) $g(x+y) \le g(x) + g(y)$ for all $x, y \in X$.
Let $M = \{x : g(x) = 0\}$. Show that the function

$$\|[x]\| := \inf_{m \in M} g(x+m)$$

provides a **norm** for the space X/M (the quotient space of X modulo M – see section 2.14), and hence that $(X/M, \|\cdot\|)$ is a normed vector space.
[This is Problem 19 on page 45 of the text.]

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