ECE / CS 541 Midterm

October 19, 2016

This is an in-class exam, 75 minutes is alloted for its completion. You are permitted one standard page crib sheet with writing on both sides. Submit your crib sheet along with your solutions.

The exam has 100 points. It is doable in 75 minutes by a student who is master of the material. The suggested solution strategy is to look over the exam and address the problems for which you have most confidence first.

Show all of your work on these exam sheets (using backsides as needed). Write your name at the top of every page. You may include additional pages if needed, but be sure to write your name on each of these and be clear how they connect to the solution on the exam pages.

Question	Maximum Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	15	
7	25	

Name ____

Name ________________________________

1. **(10 pts)** Imagine an experiment involving a blue coin and a red coin. Neither coin is fair. The red coin has probability of heads equal to 0.4, the blue coin has probability of heads equal to 0.8. Each coin toss is independent of every other.

   We start a sequence of coin tosses, alternating between coins. The first coin toss uses the red coin, the second coin toss uses the blue coin, the third toss uses the red coin, and so on. We stop once heads has been tossed twice in succession, e.g., T(red),H(blue),T(red),T(blue),H(red),T(blue),H(red),H(blue).

   If K is the number of tosses until the second successive head is encountered (in the above $K = 8$), then develop the probability mass function for K, i.e., develop a formula for $P(K = n)$, for $n = 1, 2, ...$

   Suggestion: consider the cases of K even and K odd separately.

Name ________________________________

2. **(10 pts)** Prove without using a Venn diagram that if A and B are events such that $A \subset B$, then for any event C

$$P(A \cap C) \leq P(B \cap C)$$

Name ________________________________

3. **(15 pts)** Give the name of the probability distribution associated with each of the following random variables and identify the distribution's parameter(s)

   (a) Outcome of coin flip when seeing heads defines value 1, and seeing tails defines values 0, with probability p of seeing a heads.

   (b) For N times toss a coin that has probability p of heads. The random variable is the number of tosses that show heads.

   (c) Number of coin tosses to reveal the first head, with probability of tossing a head being p.

   (d) Given $T > 0$, the largest n such that $\sum_{i=1}^{n} X_i < T$ where the X_i are independent and identically and distributed exponentials with rate λ.

   (e) $\sum_{i=1}^{N} X_i$ where N is geometric with success parameter p, and the X_i are independent and identically and distributed exponentials with rate λ .

Name ________________________________

4. **(10 pts)** Three die are tossed, each toss is independent of the others, and each toss, each die face has probability (1/6) of appearing. Derive the probability that no die face has value greater than four, *and* exactly one die face is odd, given that the sum of faces is 9.

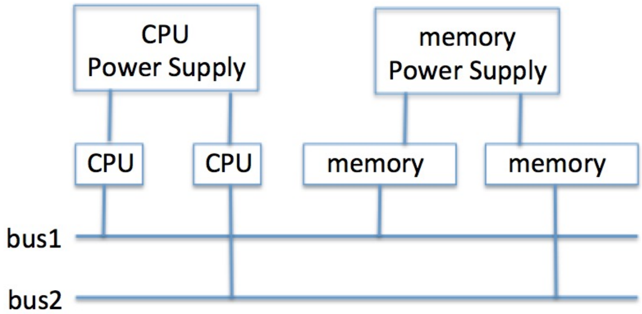
Name ______________________________

5. **(15 pts)** Consider the computer architecture in the figure below. The system is operational if either bus has an operational CPU and operational memory bank attached to it. Failure of the CPU power supply fails both CPUs; failure of the memory power supply fails both memories. A CPU fails at rate λ_p, a memory module fails at rate λ_m, a power supply (either CPU or memory) fails at rate λ_s. Assume that the buses do not fail.

   (a) Draw a reliability block diagram for this system.

   (b) Draw a fault-tree for this system.

   (c) Assuming that the processor, memory, and power-supply failure times are exponentially distributed, write an equation giving the cumulative distribution function of the reliability function, i.e., an equation for $R(t) = \Pr\{\text{System fails by time } t\}$.



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6. **(15 pts)** Consider a web server where requests arrive at rate λ_a. The server can handle up to 4 requests at a time, so an arrival when the server has 4 active requests is rejected.

   The server has one CPU, and the service rate of the CPU is λ_s (which means a job's mean time in service is $1/\lambda_s$).

   The server can fail, when there are k active requests the failure rate is $\lambda_f^{\log(k+2)}$. On failure, all active requests are lost, a repair process is initiated that completes with rate λ_r, and when in the repair state all incoming requests are rejected. Upon completion of the repair the server will accept requests again, but has no backlog.

   The CTMC has 6 states. State k reflects that the server is not being repaired, and there are k active requests. State 5 reflects the server being in the "repair" state.

   (a) Draw a state diagram for this CTMC.

   Let π_i denote the stationary probability of this CTMC being in state i. Develop formula for

   (b) the fraction of arrivals that are rejected, for any reason.

   (c) the availability of the server.

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7. **(25 pts)** Circle T (True) or F (False) on each statement below.

   T F If P is an $n \times n$ stochastic matrix (meaning every element is non-negative, and for every row the sum of elements is 1), then there is a unique solution to the equation $\pi = \pi P$.

   T F Suppose Q is an $n \times n$ transition matrix for a continuous time Markov chain, and λ_1^* and λ_2^* are positive constants such that $\lambda_1^* < \lambda_2^*$ and $\lambda_1^* \geq \max_{1 \leq i \leq n}\{-q_{i,i}\}$. For any $0 < \epsilon < 1$ let $N_1(\epsilon)$ be the largest number of DTMC transitions we compute using λ_1^* as the uniformization constant to achieve a per-term error bound of ϵ; likewise let $N_2(\epsilon)$ be that number when we use λ_2^*. Then $N_1(\epsilon) \leq N_2(\epsilon)$.

   T F Every continuous time Markov chain can be uniformized.

   T F The "embedded" discrete time Markov chain associated with a continuous time Markov chain has the property that for any state i, the probability of returning to state i eventually is 1.

   T F Every discrete-time Markov chain with a finite number of states has stationary state-occupancy probabilities, i.e., for every state i the limit below exists

$$\lim_{k \to \infty} \Pr\{\text{being in state } i \text{ after exactly } k \text{ transitions}\} = \pi_i, \text{for some } \pi_i.$$