

# Lecture 13: Concatenated Tube Models

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ECE 537: Speech Processing

- 1 Review: Acoustic waves in a lossless tube
- 2 Concatenated Tube Model
- 3 Flow and pressure continuity at the boundary between two tubes
- 4 Discrete-time implementation
- 5 Example
- 6 Summary

# Outline

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# Constitutive equations

In a lossless tube we have:

- Mass continuity:

$$-\frac{\partial u}{\partial x} = \frac{\partial A}{\partial t} + \frac{A}{\rho c^2} \frac{\partial p}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A} \frac{\partial u}{\partial t}$$

- Wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

# Solution: Forward-going and backward-going waves

Any combination of a forward-going wave plus a backward-going wave is a solution to the wave equation:

$$u(x, t) = u_+(t - x/c) - u_-(t + x/c)$$

$$p(x, t) = \frac{\rho c}{A} (u_+(t - x/c) + u_-(t + x/c))$$

The Fourier transforms of  $u(x, t)$  and  $p(x, t)$  are:

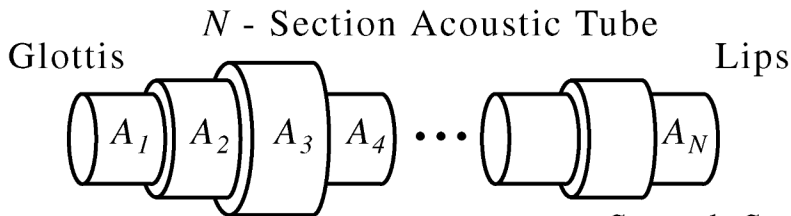
$$U(x, \Omega) = U_+(\Omega)e^{-j\Omega x/c} - U_-(\Omega)e^{j\Omega x/c}$$

$$P(x, \Omega) = \frac{\rho c}{A} (U_+(\Omega)e^{-j\Omega x/c} + U_-(\Omega)e^{j\Omega x/c})$$

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# Concatenated-Tube Model (Kelly and Lochbaum, 1962)



# Concatenated-Tube Model

Let the  $i^{\text{th}}$  tube have length  $l_i$ , area  $A_i$ , forward-going wave  $u_i^+(t)$ , and backward-going wave  $u_i^-(t)$ . The flow and pressure in this tube are

$$u_i(x, t) = u_i^+(t - x/c) - u_i^-(t + x/c)$$

$$p_i(x, t) = \frac{\rho c}{A_i} (u_i^+(t - x/c) + u_i^-(t + x/c))$$

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# Flow and pressure continuity at the boundary between two tubes

Flow and pressure are continuous at the boundary between tubes  $i$  and  $i + 1$ :

$$u_i(l_i, t) = u_{i+1}(0, t)$$

$$p_i(l_i, t) = p_{i+1}(0, t)$$

Substituting forward and backward waves:

$$u_i^+(t - \tau_i) - u_i^-(t + \tau_i) = u_{i+1}^+(t) - u_{i+1}^-(t)$$

$$\frac{A_{i+1}}{A_i} (u_i^+(t - \tau_i) + u_i^-(t + \tau_i)) = (u_{i+1}^+(t) + u_{i+1}^-(t))$$

... where  $\tau_i = l_i/c$  is the time it takes a wave to traverse the  $i^{\text{th}}$  tube.

# Incident vs. Reflected Waves

$$u_i^+(t - \tau_i) - u_i^-(t + \tau_i) = u_{i+1}^+(t) - u_{i+1}^-(t)$$
$$\frac{A_{i+1}}{A_i} (u_i^+(t - \tau_i) + u_i^-(t + \tau_i)) = u_{i+1}^+(t) + u_{i+1}^-(t)$$

The waves  $u_i^+(t - \tau_i)$  and  $u_{i+1}^-(t)$  are **incident on** the boundary. The waves  $u_i^-(t + \tau_i)$  and  $u_{i+1}^+(t)$  are **reflected from** the boundary. Writing the latter in terms of the former,

$$u_{i+1}^+(t) = \left[ \frac{2A_{i+1}}{A_{i+1} + A_i} \right] u_i^+(t - \tau_i) + \left[ \frac{A_{i+1} - A_i}{A_{i+1} + A_i} \right] u_{i+1}^-(t)$$
$$u_i^-(t + \tau_i) = - \left[ \frac{A_{i+1} - A_i}{A_{i+1} + A_i} \right] u_i^+(t - \tau_i) + \left[ \frac{2A_i}{A_{i+1} + A_i} \right] u_{i+1}^-(t)$$

# Reflection Coefficients

We can simplify the reflection equations by defining a reflection coefficient:

$$r_i = \frac{A_{i+1} - A_i}{A_{i+1} + A_i}$$

Then:

$$\begin{aligned} u_{i+1}^+(t) &= (1 + r_i)u_i^+(t - \tau_i) + r_i u_{i+1}^-(t) \\ u_i^-(t + \tau_i) &= -r_i u_i^+(t - \tau_i) + (1 - r_i)u_{i+1}^-(t) \end{aligned}$$

$$r_i = \frac{A_{i+1} - A_i}{A_{i+1} + A_i}$$


# Reflection at the Lips

The  $N^{\text{th}}$  tube connects at the lips, with reflection coefficient

$$r_N = \frac{\frac{\rho c}{A_N} - Z_L}{\frac{\rho c}{A_N} + Z_L}$$

... where  $Z_L$  is the radiation impedance,  $P_L(\Omega) = Z_L(\Omega)U_L(\omega)$ .  
The reflected wave is

$$u_N^-(t + \tau_N) = -r_N u_N^+(t - \tau_N)$$

... and the volume velocity of air leaving the lips is

$$u_N(l_N, t) = u_N^+(t - \tau_i) - u_N^-(t + \tau_i)$$

# Reflection at the Glottis

Similar to the lips, some flow might be transmitted backward through the glottis with  $P_G(\Omega) = Z_G(\Omega)U_G(\Omega)$  and

$$r_0 = \frac{Z_G - \frac{\rho c}{A_1}}{Z_G + \frac{\rho c}{A_1}}$$

The forward-going wave in the first tube includes the glottal source  $u_G(t)$  and the reflection of the backward-going wave:

$$u_1^+(t) = (1 + r_0)u_G(t) + r_0u_1^-(t)$$

# Summary: Concatenated-Tube Model

The impedance of a tube is  $Z_i = \frac{\rho c}{A_i}$ , and

$$\begin{aligned} r_i &= \frac{A_{i+1} - A_i}{A_{i+1} + A_i} \\ &= \frac{Z_i - Z_{i+1}}{Z_i + Z_{i+1}} \end{aligned}$$

The scattering equations at each boundary are:

$$\begin{aligned} u_{i+1}^+(t) &= (1 + r_i)u_i^+(t - \tau_i) + r_i u_{i+1}^-(t) \\ u_i^-(t + \tau_i) &= -r_i u_i^+(t - \tau_i) + (1 - r_i)u_{i+1}^-(t) \end{aligned}$$

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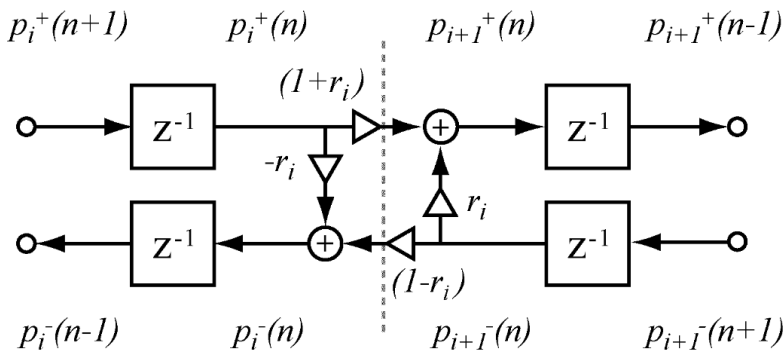
# Discrete-time implementation

Suppose each tube has the same length, such that  $\tau_i = \frac{1}{F_s}$  is one sampling time. Then the discrete-time implementation is

$$\begin{aligned}u_{i+1}^+[n] &= (1 + r_i)u_i^+[n-1] + r_i u_{i+1}^-[n] \\ u_i^-[n+1] &= -r_i u_i^+[n-1] + (1 - r_i)u_{i+1}^-[n]\end{aligned}$$

# Discrete-time implementation

Using  $z^{-1}$  to express a one-sample delay, we get:



# Discrete-time implementation: Half the sampling rate

We can make this even more efficient if we use shorter tubes,  $\tau_i = \frac{1}{2F_s}$ , only half of one sample time. Then the discrete-time implementation is

$$\begin{aligned}u_{i+1}^+[n] &= (1 + r_i)u_i^+[n - \frac{1}{2}] + r_i u_{i+1}^-[n] \\ u_i^-[n + \frac{1}{2}] &= -r_i u_i^+[n - \frac{1}{2}] + (1 - r_i)u_{i+1}^-[n]\end{aligned}$$

Define  $f_i[n] = u_i^+[n + \frac{i}{2}]$  and  $b_i[n] = u_i^-[n + \frac{i}{2}]$ , then we have

$$\begin{aligned}f_{i+1}[n] &= (1 + r_i)f_i[n - 1] + r_i b_{i+1}[n] \\ b_i[n] &= -r_i f_i[n - 1] + (1 - r_i)b_{i+1}[n]\end{aligned}$$

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# Example

In the last lecture, we considered two-tube models in which one of the two tubes had much lower area than the other. What is the reflection coefficient at the juncture between two such tubes?  
What are the equivalent boundary conditions for pressure and flow?

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# Summary

$$\begin{aligned}
 r_i &= \frac{A_{i+1} - A_i}{A_{i+1} + A_i} \\
 &= \frac{Z_i - Z_{i+1}}{Z_i + Z_{i+1}}
 \end{aligned}$$

