

Lecture 12: Two-Tube Models

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ECE 537: Speech Processing

- 1 Review: Acoustic waves in a lossless tube
- 2 Other one-tube resonators
- 3 Two-tube resonators
- 4 Helmholtz resonators
- 5 Example
- 6 Summary

Outline

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Constitutive equations

In a lossless tube we have:

- Mass continuity:

$$-\frac{\partial u}{\partial x} = \frac{\partial A}{\partial t} + \frac{A}{\rho c^2} \frac{\partial p}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A} \frac{\partial u}{\partial t}$$

Wave equation in a lossless uniform tube

The wave equation can be written in terms of either pressure or flow:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$

Solution: Forward-going and backward-going waves

Any combination of a forward-going wave plus a backward-going wave is a solution to the wave equation:

$$u(x, t) = u_+(t - x/c) - u_-(t + x/c)$$

$$p(x, t) = \frac{\rho c}{A} (u_+(t - x/c) + u_-(t + x/c))$$

The Fourier transforms of $u(x, t)$ and $p(x, t)$ are:

$$U(x, \Omega) = U_+(\Omega)e^{-j\Omega x/c} - U_-(\Omega)e^{j\Omega x/c}$$

$$P(x, \Omega) = \frac{\rho c}{A} (U_+(\Omega)e^{-j\Omega x/c} + U_-(\Omega)e^{j\Omega x/c})$$

Quarter-wave resonator with zero input

A closed glottis has zero flow. Open lips permit zero pressure:

$$U(0, \Omega) = 0$$

$$P(l, \Omega) = 0$$

The solution is nonzero only at formant frequencies

$$F_{k+1} = \frac{c}{4l} + \frac{kc}{2l}:$$

$$U_+(\Omega) = -U_+(\Omega)e^{2j\Omega l/c}$$

$$U_+(\Omega) \cos(\Omega l/c) = 0$$

Quarter-wave resonator with zero input

In a uniform tube closed at one end and open at the other (i.e., / $\mathfrak{a}$ /):

$$U(0, \Omega) = 0$$

$$P(l, \Omega) = 0$$

$$U_+(\Omega) \cos(\Omega l / c) = 0$$

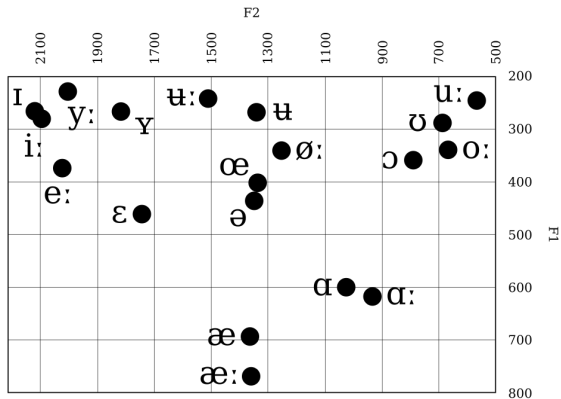
...so $U_+(\Omega)$ can only be nonzero at the frequencies

$$\Omega_k = \frac{\pi c}{2l} + (k-1) \frac{\pi c}{l}$$

$$F_k = \frac{c}{4l} + (k-1) \frac{c}{2l}$$

i.e., the frequencies $F_k = 500, 1500, 2500, 3500, \dots$

Schwa formant frequencies: 500, 1500, ...



Quarter-wave resonator with fixed input

Taking into account glottal flow, we have:

$$U(0, \Omega) = U_G(\Omega)$$

$$P(l, \Omega) = 0$$

The solution is infinite at formant frequencies $F_{k+1} = \frac{c}{4l} + \frac{kc}{2l}$:

$$U_+(\Omega) = U_G(\Omega) \frac{e^{j\Omega l/c}}{2 \cos(\Omega l/c)}$$

$$U_-(\Omega) = -U_G(\Omega) \frac{e^{-j\Omega l/c}}{2 \cos(\Omega l/c)}$$

Quarter-wave resonator with losses

The most important losses are radiation losses through the glottis and lips:

$$U(0, \Omega) = U_G(\Omega) + \frac{1}{Z_G(\Omega)} P(0, \Omega)$$

$$P(l, \Omega) = Z_R(\Omega) U(l, \Omega)$$

The solution has big peaks at the formant frequencies, governed by some loss term $\sigma(\Omega)$:

$$U_+(\Omega) = U_G(\Omega) \frac{e^{j\Omega l/c}}{\sigma(\Omega) + 2 \cos(\Omega l/c)}$$

$$U_-(\Omega) = -U_G(\Omega) \frac{e^{-j\Omega l/c}}{\sigma(\Omega) + 2 \cos(\Omega l/c)}$$

Summary of losses in the vocal tract

Putting it all together, the formant frequencies and bandwidths of a schwa (/ə/) are approximately

	Freq	BW
F1	500	100
F2	1500	200
F3	2500	300
F4	3500	400

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Tube closed at both ends: Half-wave resonator

Consider the vowels /u,o,v/, in which the glottis is nearly closed, and the lips are nearly closed:

$$U(0, \Omega) = 0$$

$$U(l, \Omega) = 0$$

The general solution

$$U(x, \Omega) = U_+(\Omega)e^{-j\Omega x/c} - U_-(\Omega)e^{j\Omega x/c}$$

Is solved to give

$$U_+(\Omega) = U_+(\Omega)e^{2j\Omega l/c}$$

$$U_+(\Omega) \sin(\Omega l/c) = 0$$

Tube closed at both ends: Half-wave resonator

$$U(0, \Omega) = 0$$

$$U(l, \Omega) = 0$$

$$U_+(\Omega) \sin(\Omega l/c) = 0$$

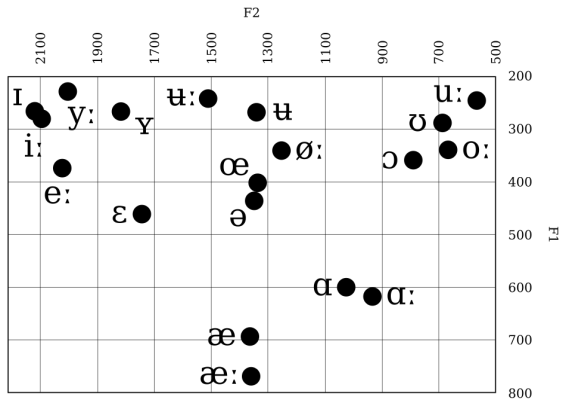
In a tube closed at both ends, $U_+(\Omega)$ can only be nonzero at the frequencies

$$\Omega_k = (k-1) \frac{\pi c}{l}$$

$$F_k = (k-1) \frac{c}{2l}$$

i.e., the frequencies $F_k = 0, 1000, 2000, 3000, \dots$

Formants of /u,ʊ,o/: 0, 1000,...



Tube open at both ends: Half-wave resonator

Suppose we had a tube open at both ends! The glottis can't open that wide, but some flutes are open at both ends. In that case:

$$P(0, \Omega) = 0$$

$$P(l, \Omega) = 0$$

The general solution

$$P(x, \Omega) = \frac{\rho c}{A} \left(U_+(\Omega) e^{-j\Omega x/c} + U_-(\Omega) e^{j\Omega x/c} \right)$$

Is solved to give

$$U_+(\Omega) = U_+(\Omega) e^{2j\Omega l/c}$$

$$U_+(\Omega) \sin(\Omega l/c) = 0$$

Tube open at both ends: Half-wave resonator

$$P(0, \Omega) = 0$$

$$P(l, \Omega) = 0$$

$$U_+(\Omega) \sin(\Omega l / c) = 0$$

In a tube open at both ends, $U_+(\Omega)$ can only be nonzero at the frequencies

$$\Omega_k = (k - 1) \frac{\pi c}{l}$$

$$F_k = (k - 1) \frac{c}{2l}$$

i.e., the frequencies $F_k = 0, 1000, 2000, 3000, \dots$

One-Tube Resonators: Summary

- Schwa (/ə/) can be approximated by a tube closed at one end ($U(0, \Omega) \approx 0$) and open at the other ($P(l, \Omega) \approx 0$). This is a **quarter-wave resonator**, so its formant frequencies are

$$F_k = \frac{c}{4l} + (k-1)\frac{c}{2l}$$

- Rounded vowels (/u, ʊ, o/) can be approximated by a tube closed at both ends ($U(0, \Omega) = U(l, \Omega) = 0$; we'll learn a better approximation in the homework). Tubes closed at both ends or open at both ends are **half-wave resonators**, so their formant frequencies are

$$F_k = (k-1)\frac{c}{2l}$$

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Two-Tube Resonators

- Low vowels ($/a, \Lambda, \text{æ}, \text{ɛ}/$) can be approximated by a two-tube model with a constricted pharynx and an open mouth.
- High front vowels ($/i, \text{ɪ}, \text{e}/$) can be approximated by a two-tube model with a constricted mouth and an open pharynx.
- Lecture 13 will study interaction between the tubes, but two-tube models work pretty well even if you assume no interaction.
- See https://www.phon.ox.ac.uk/tube_models for visualizations.

Two-Tube Model: Low vowels

- Suppose the area of the glottis is less than the area of the pharynx, which is less than the area of the mouth:
 $A_g \ll A_1 \ll A_2$.
- Both tubes are closed at the left, and open at the right, so both are **quarter-wave resonators**.
- The formant frequencies include resonances of both tubes:

$$F_k \in \left\{ \frac{c}{4l_1}, \frac{c}{4l_2}, \frac{3c}{4l_1}, \frac{3c}{4l_2}, \frac{5c}{4l_1}, \dots \right\}$$

Two-Tube Model: Low vowels

For example, consider a very low back vowel with

$$c = 354 \frac{m}{s}$$

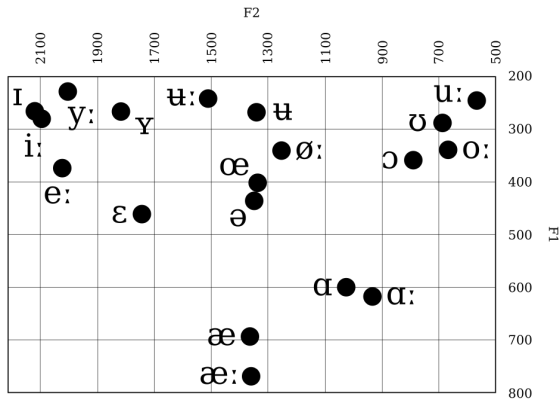
$$l_1 = 0.177m$$

$$l_2 = 0.177m,$$

then

$$F_k \in \{1000, 1000, 3000, 3000, 5000, \dots\}$$

Formants of /ɑ/: $F_1 \approx 1000, F_2 \approx 1000$



Two-Tube Model: High front vowels

- Suppose the tongue is pushed up into the mouth, so that the pharynx is wide open: $A_g \ll A_2 \ll A_1$.
- The pharynx tube is mostly closed at both ends, so it's a half-wave resonator.
- The mouth tube is mostly open at both ends, so it's also a half-wave resonator!
- The formant frequencies include resonances of both tubes:

$$F_k \in \left\{ 0, \frac{c}{2l_1}, \frac{c}{2l_2}, \frac{c}{l_1}, \frac{c}{l_2}, \frac{3c}{2l_1}, \dots \right\}$$

Two-Tube Model: High front vowels

For example, consider a very high front vowel with

$$c = 354 \frac{m}{s}$$

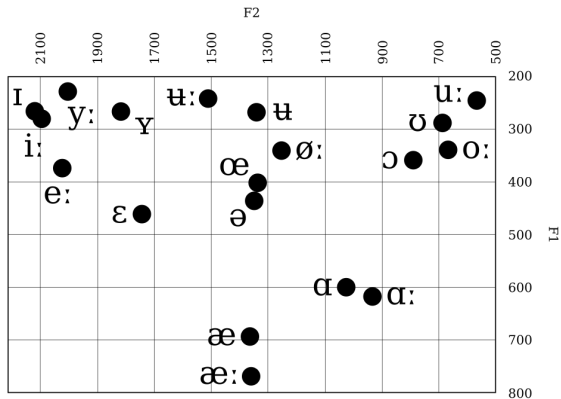
$$l_1 = 0.177m$$

$$l_2 = 0.177m,$$

then

$$F_k \in \{0, 2000, 2000, 4000, 4000, 6000, \dots\}$$

Formants of /ɪ,i,e/: $F_1 \approx 0, F_2 \approx 2000$



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What does it mean to have a resonance at 0Hz?

- If the tube is closed at both ends, a 0Hz resonance means that you pump up the air pressure inside, like a balloon.
- If the tube is open at both ends, a 0Hz resonance means you are blowing air through the tube at a constant rate with no variation.
- Real high vowels, like /i,ɪ,e,u,ʊ,o/, have their first formant not at 0Hz, but at $200 \leq F_1 \leq 500$ Hz. Why?

Why is F_1 not quite zero?

- Rounded vowels (/u,ʊ,o/) have lips not quite closed. High oral pressure accelerates air forward through the lips; low oral pressure accelerates air backward through the lips.
- High front vowels (/i,I,e/) have tongue body not quite closed. High pharynx pressure accelerates air forward through the mouth; low pharynx pressure accelerates air backward through the mouth.

Helmholtz resonator model: Front tube is an acoustic mass

A **Helmholtz resonator** is a two-tube model with an open back cavity and a narrow front cavity. Air pressure drop across the front tube accelerates air through it:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A_2} \frac{\partial u}{\partial t}$$

If the back cavity air pressure is $p_1(t)$, and if we assume that air pressure drops linearly to zero over the length l_2 , then

$$\frac{\partial p_2(x, t)}{\partial x} = \frac{p_1(t)}{l_2}$$

... and therefore:

$$p_1(t) = \frac{\rho l_2}{A_2} \frac{\partial u_2(t)}{\partial t}$$

Helmholtz resonator model: Back tube is an acoustic compliance

Remember the mass continuity equation:

$$-\frac{\partial u}{\partial x} = \frac{\partial A}{\partial t} + \frac{A}{\rho c^2} \frac{\partial p}{\partial t}$$

If the glottis has zero airflow, the front cavity has nonzero flow $u_2(t)$, and the airflow increases from 0 to $u_2(t)$ linearly over the length l_1 , then

$$\frac{\partial u_1(x, t)}{\partial x} = \frac{u_2(t)}{l_1}$$

... and therefore (assuming $\partial A / \partial t = 0$):

$$-u_2(t) = \frac{A_1 l_1}{\rho c^2} \frac{\partial p_1}{\partial t}$$

Helmholtz resonator model: Time domain equations

Putting it together, we can treat the narrow front cavity as a lumped acoustic mass, while the back cavity is a lumped acoustic compliance:

$$p_1(t) = M_2 \frac{\partial u_2}{\partial t}$$

$$u_2(t) = -C_1 \frac{\partial p_1}{\partial t}$$

... where ...

$$M_2 = \frac{\rho l_2}{A_2}$$

$$C_1 = -\frac{V_1}{\rho c^2}$$

... where $V_1 = A_1 l_1$ is the volume of the back cavity.

Helmholtz resonator model: Frequency domain equations

Taking the Fourier transforms,

$$P_1(\Omega) = j\Omega M_2 U_2(\Omega)$$

$$U_2(\Omega) = -j\Omega C_1 P_1(\Omega)$$

... where ...

$$M_2 = \frac{\rho l_2}{A_2}$$

$$C_1 = -\frac{V_1}{\rho c^2}$$

Helmholtz resonator model: Resonance equations

Combining the two equations, we find that:

$$P_1(\Omega) (1 - \Omega^2 M_2 C_1) = 0$$

...so, if there is no excitation (no input from the glottis), $P_1(\Omega)$ can be nonzero only at the formant frequency, which is

$$\Omega = \frac{1}{\sqrt{M_2 C_1}} = c \sqrt{\frac{A_2}{l_2 V_1}}$$

$$M_2 = \frac{\rho l_2}{A_2}$$

$$C_1 = -\frac{V_1}{\rho c^2}$$

Helmholtz resonators of real vowels

High and mid vowels (/i,ɪ,e,u,ʊ,o/) have $\Omega_1 = 2\pi F_1$ in the range of $2\pi 200 \leq \Omega_1 \leq 2\pi 500$:

$$2\pi 200 [s^{-1}] \leq c \sqrt{\frac{A_2}{l_2 V_1}} \leq 2\pi 500 [s^{-1}]$$

If we approximate $c \approx 100\pi [\frac{m}{s}]$, $l_1 \approx 0.1 [m]$, and $l_2 \approx 0.1 [m]$, then the area ratio is approximately bounded as

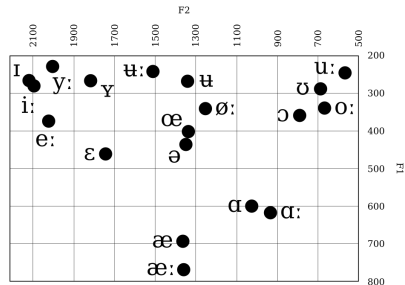
$$6.3 \geq \frac{A_1}{A_2} \geq 1$$

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Examples

- What might be l_1 and l_2 for /a,æ/?
- What might be l_2 for /i,i,e/?



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Summary

- Quarter-wave resonator: Closed at one end, open at the other.
- Half-wave resonators: Closed at both ends or open at both ends.
- Low vowel = two quarter-wave resonators.
- High front vowel = two half-wave resonators.
- Helmholtz resonator:

$$\Omega = \frac{1}{\sqrt{C_1 M_2}} = c \sqrt{\frac{A_2}{l_2 V_1}}$$