Lecture 11: Acoustic Wave Equations

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ECE 537: Speech Processing

Outline

Mass of Air

- Define the "area function" of the vocal tract to be A ($A(x,t)$ when we are listing arguments). It's a function of distance from the glottis (x) and time (t).
- $\rho(x,t)$ is the density of air.
- The mass of an incremental volume of air at position x is:

$$\rho A \partial x$$

Volume velocity and mass velocity

- Define $u(x,t)$ to be the total volume of air flowing through that location (measured in m^3/s). We call this "volume velocity."
- We could define the mass velocity (in kg/s) to be

$$\rho u$$

What if $u(x,t)$ depends on x?

- If $u(x,t)$ depends on x, then the mass of the air at position x must be changing. Suppose that $\partial u/\partial t$ is negative (i.e., there is some downstream "blockage" of the flow), then we have that:

$$-\frac{\partial \rho u}{\partial x} = \frac{\partial \rho A}{\partial t}$$

- Experiments suggest that $\partial\rho/\partial x$ is small enough to be neglected, but $\partial\rho/\partial t$ is not, so:

$$-\rho\frac{\partial u}{\partial x} = \rho\frac{\partial A}{\partial t} + A\frac{\partial \rho}{\partial t}$$

  In words: "blockage of flow results in expansion of the tube or increase of density."

The ideal gas law

- The relationship between pressure, volume, mass, and temperature in a gas is often written as:

  $$pV = nRT$$

  …where p is pressure, V is volume, n is the number of moles (mass divided by molecular weight), R is the ideal gas constant, and T is temperature.
- In acoustics we notice that $rho \propto n/V$, so

  $$p \propto \rho T$$

Isothermal vs. Adiabatic Expansion

- If you change the volume of a fluid slowly enough, then all of the generated heat has time to leak away, so $p \propto \rho T$ becomes $p \propto \rho$. This is isothermal expansion.
- On the other hand, if change is fast, there is no time for heat to leak away. This case is called **lossless** because there are no heat losses; it is also called **adiabatic**, meaning "with no loss of heat." In this case, T and ρ both change in the same direction as pressure changes. The linearized version of this equation is

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial \rho}{\partial t}$$

  where c is a constant of proportionality called the "speed of sound" that varies quite a lot depending on temperature and pressure.

Newton's second law in a lossless tube

- Newton's second law is

$$f = ma$$

- In the lossless case, there are no viscous forces, so $f = -A\partial p$. The mass and acceleration are $m = \rho A \partial x$, and $a = (1/A)\partial u/\partial t$, so

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A}\frac{\partial u}{\partial t}$$

Constitutive equations

In a lossless tube we have:

- Mass continuity:

$$-\rho\frac{\partial u}{\partial x} = \rho\frac{\partial A}{\partial t} + A\frac{\partial \rho}{\partial t}$$

- Adiabatic ideal gas law:

$$\frac{\partial p}{\partial t} = c^2\frac{\partial \rho}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A}\frac{\partial u}{\partial t}$$

Combined constitutive equations

We often combine the constitutive equations to get just two equations in terms of the variables $p(x,t)$ and $u(x,t)$. This is the simplest they get without extra assumptions:

- Mass continuity:

$$-\frac{\partial u}{\partial x} = \frac{\partial A}{\partial t} + \frac{A}{\rho c^2}\frac{\partial p}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A}\frac{\partial u}{\partial t}$$

Outline

Lossless uniform tube

Suppose we assume that $A(x,t) = A_0$, a fixed area. Then $\partial A/\partial t = 0$, so

- Mass continuity:

$$-\frac{\partial u}{\partial x} = \frac{A}{\rho c^2}\frac{\partial p}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A}\frac{\partial u}{\partial t}$$

Wave equation in a lossless uniform tube

Combining the two equations, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

…and once we solve for $u(x,t)$, we can find $p(x,t)$ using either of the two constitutive equations.

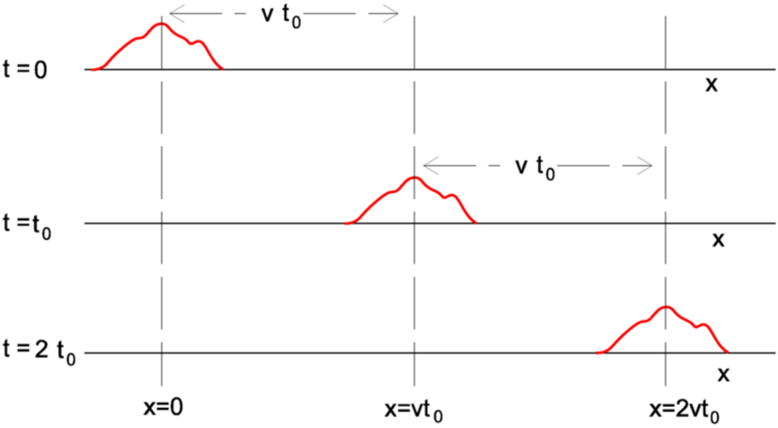
Solution: Any forward-going wave, added to any backward-going wave

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Any combination of a forward-going wave plus a backward-going wave is a solution to this equation:

$$u(x,t) = u_+(t - x/c) - u_-(t + x/c)$$
$$p(x,t) = \frac{\rho c}{A}\left(u_+(t - x/c) + u_-(t + x/c)\right)$$

Solution example: Forward-going wave



Fourier series solution

Suppose $u_+(t)$ and $u_-(t)$ are periodic, with Fourier series coefficients $U_+(\Omega)$ and $U_-(\Omega)$ at some frequency Ω. The wave behavior at that frequency is

$$
\begin{aligned}
u(x,t) &= U_+(\Omega)^{j\Omega(t-x/c)} - U_-(\Omega)e^{j\Omega(t+x/c)} \\
p(x,t) &= \frac{\rho c}{A}\left(U_+(\Omega)e^{j\Omega(t-x/c)} + U_-(\Omega)e^{j\Omega(t+x/c)}\right)
\end{aligned}
$$

Outline

Boundary conditions at the glottis

At the glottis ($x = 0$), the boundary condition is a known amount of air coming into the vocal tract from the glottis:

$$u_G(t) = u(0,t)$$
$$= u_+(t) - u_-(t)$$
$$U_G(\Omega) = U_+(\Omega)^{j\Omega t} - U_-(\Omega)e^{j\Omega t}$$

Solution: the forward-going wave equals the reflection of the backward-going wave, plus whatever is added by the glottis:

$$U_+(\Omega) = U_G(\Omega) + U_-(\Omega)$$

Boundary conditions at the lips

At the glottis ($x = l$), the air pressure is very nearly constant at atmospheric pressure. The non-constant part is $p(l,t) \approx 0$, so

$$0 = \frac{\rho c}{A}\left(U_+(\Omega)e^{j(t-l/c)} + U_-(\Omega)e^{j(t+l/c)}\right)$$

Solution: volume velocity arriving at the lips is reflected back, delayed by $2l/c$, the time it takes to go forward and back again:

$$U_-(\Omega) = -U_+(\Omega)e^{-j2l/c}$$

Solution: Lossless uniform tube

Solving for $U_+(\Omega)$ and $U_-(\Omega)$ in terms of $U_G(\Omega)$ gives:

$$U_+(\Omega) = U_G(\Omega)\frac{1}{1+e^{-j\Omega 2l/c}}$$

$$U_-(\Omega) = -U_G(\Omega)\frac{e^{-j\Omega 2l/c}}{1+e^{-j\Omega 2l/c}}$$

Solution: Standing wave pattern

Solving for $u(x, t)$ gives:

$$\begin{aligned} u(x,t) &= U_+(\Omega)^{j\Omega(t-x/c)} - U_-(\Omega)e^{j\Omega(t+x/c)} \\ &= U_G(\Omega)\frac{\cos(2\Omega(l-x)/c)}{\cos(2\Omega l/c)}e^{j\Omega t} \end{aligned}$$

- When the input is in sinusoidal steady state $U_G(\Omega)e^{j\Omega t}$, the whole system enters sinusoidal steady state.
- The result is a **standing wave** pattern: $u(x,t) = u(x)e^{j\Omega t}$, where $u(x)$, the standing wave, doesn't depend on t (and the time dependence, $e^{j\Omega t}$, doesn't depend on x).

Solution: Quarter-wave resonator

Wikipedia doesn't have any image examples of a quarter-wave resonator, but you can imagine it as just the left half of this image: https://commons.wikimedia.org/wiki/File:Dipole_antenna_standing_waves_animation_1-10fps.gif

Solution: Quarter-wave resonator

$$u(x,t) = U_G(\Omega)\frac{\cos(2\Omega(l-x)/c)}{\cos(2\Omega l/c)}e^{j\Omega t}$$

- This particular standing wave has an odd feature: If $U_G(\Omega) \neq 0$, then $u(x,t) = \infty$ at

$$\Omega = \frac{\pi c}{2l} + k\frac{\pi c}{l} \quad \text{for any integer } k$$

- This is called a quarter-wave resonator because, using $\lambda = 2\pi c/\Omega$, we get

$$l = \frac{\lambda}{4} \pm k\frac{\lambda}{2} \quad \text{for any integer } k$$

- In a real tube, losses $\Rightarrow$ finite $u(x,t)$, but it might still be much larger than $u_G(t)$ at those frequencies.

Solution: Quarter-wave resonator

$$u(x,t) = U_G(\Omega)\frac{\cos(2\Omega(l-x)/c)}{\cos(2\Omega l/c)}e^{j\Omega t}$$

$$p(x,t) = U_G(\Omega)\frac{j\sin(2\Omega(l-x)/c)}{\cos(2\Omega l/c)}e^{j\Omega t}$$

At the standing wave frequencies $\Omega = \frac{\pi c}{2l} + k\frac{\pi c}{l}$,

- Zero flow at the glottis corresponds to maximum flow at the lips (and nonzero glottal flow corresponds to infinite lip flow),
- Zero pressure at the lips corresponds to maximum pressure at the lips

Solution: Quarter-wave resonator

In fact, we can solve for the resonant frequencies more easily by asking: at what frequencies can we have $u_G(t) = 0$ and $p_L(t) = 0$, while still having nonzero $u(x,t)$ and $p(x,t)$ elsewhere? Those will be the nonzero values of $U_+(\Omega)$ and $U_-(\Omega)$ that solve

$$
\begin{aligned}
0 &= U_+(\Omega) - U_-(\Omega) \\
0 &= \frac{\rho c}{A}\left(U_+(\Omega)e^{-j\Omega l/c} + U_-(\Omega)e^{j\Omega l/c}\right)
\end{aligned}
$$

...which has nonzero solutions only if

$$
l = \frac{\lambda}{4} + k\frac{\lambda}{2}
$$

Solution: Quarter-wave resonator

Another way to write the resonant frequencies is

$$f = \frac{c}{2l} + k\frac{c}{l}$$

The speed of sound is $c = 340 m/s$ at room temperature, but at body temperature it's more like $c = 354 m/s$. At that speed, $l/c = 1000$Hz if $l = 17.7$cm, which is possible for a person with a large head. So for a person with a large head producing a vowel with no constrictions (e.g., [ə]),

$$f = 500 + 1000k \text{ Hz}$$

Outline

Losses at the lips

Of course, just as $u_G(t)$ is not exactly zero, $p_L(t)$ is also not exactly zero. The true equation is

$$P_L(\Omega) = Z_R(\Omega) U_L(\Omega)$$

...where $Z_L(\Omega)$ is the radiation impedance.

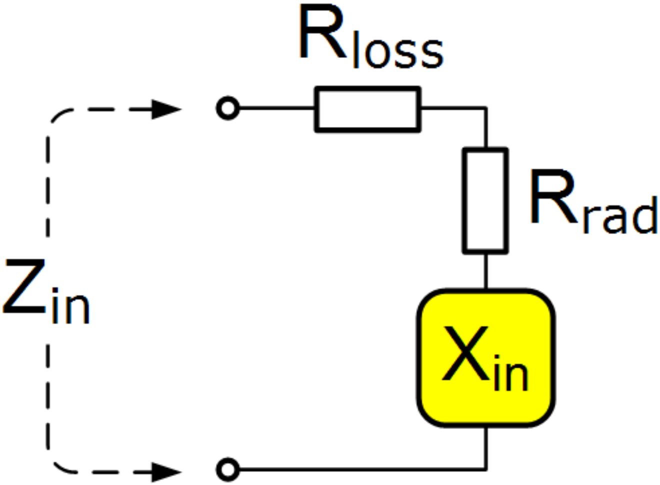
Losses at the lips

$$Z_L(\Omega) \approx R_L(\Omega) + j\Omega X_L(\Omega)$$

- $X_L(\Omega)$ is the acoustic mass (inertia) of the air just in front of the lips that needs to be accelerated in order to start a traveling wave that leaves your lips.
- $R_L(\Omega)$ is air that actually leaves your lips and travels away (relatively little compared to the standing wave amplitude, but still nonzero).

Losses at the lips

Coincidentally, acoustic radiation losses have the same general form as electromagnetic radiation losses:



Summary of losses in the vocal tract

- Losses that increase at higher frequencies: radiation losses at the lips
- Losses that are highest at low frequencies:
  - Radiation losses at the glottis (backward wave travels through glottis to the lungs)
  - Vibrating walls of the vocal tract
  - Thermal losses

Summary of losses in the vocal tract

Putting it all together, the resonant frequencies have bandwidths of approximately:

	Freq	BW
F1	500	100
F2	1500	200
F3	2500	300
F4	3500	400

Outline

Example

Solve the boundary conditions at the lips and glottis to get the transfer function for a vocal tract that's closed at the glottis and open at the lips.

Outline

Summary

- Mass continuity:

$$-\frac{\partial u}{\partial x} = \frac{\partial A}{\partial t} + \frac{A}{\rho c^2}\frac{\partial p}{\partial t}$$

- Newton's second law:

$$-\frac{\partial p}{\partial x} = \frac{\rho}{A}\frac{\partial u}{\partial t}$$

- Wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

- Lossless uniform tube:

$$\begin{aligned} u(x,t) &= U_+(\Omega)^{j\Omega(t-x/c)} - U_-(\Omega)e^{j\Omega(t+x/c)} \\ &= U_G(\Omega)\frac{\cos(2\Omega(l-x)/c)}{\cos(2\Omega l/c)}e^{j\Omega t} \end{aligned}$$