Lecture 2: Random Signals

Mark Hasegawa-Johnson

ECE 537: Speech Processing

Outline

What time is zero?

The DTFT is defined to be

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

But what time is zero?

- Buddhist zero: Midnight 544BCE, Kushinagar
- Christian zero: Midnight 0CE, Bethlehem
- Unix: Midnight UTC, January 1, 1970
- Most students: whatever time I started the recording

. . . does it matter?

Unfortunately, it matters

Suppose $y[n] = x[n - n_0]$, i.e., they are the same signal, but measured with different definitions of $n = 0$.

$$Y(\omega) = X(\omega)e^{-j\omega n_0}$$

$$Y[k] = X[k]e^{-j\frac{2\pi k n_0}{N}}$$

- If $\left|\frac{2\pi n_0}{N}\right| < \pi$, the starting time of your recording can be recovered from data using **phase unwrapping**:

$$n_0 = \underset{n_0}{\operatorname{argmin}} \left|\angle Y[k] - \left(\angle Y[k-1] - \frac{2\pi n_0}{N}\right)\right|$$

- If $n_0 > 0.5N$, phase unwrapping finds, at best, an aliased start time. In that case, phase is very hard to model – random, for most practical purposes.

How do we deal with phase?

There are two standard methods for dealing with phase:

- **Metadata:** Keep careful metadata specifying how $n = 0$ is defined. The most common example: Pitch-synchronous analysis (e.g., PSOLA=pitch-synchronous overlap-and-add) defines $n = 0$ at the moment of a glottal closure event.
- **Phase-invariant features:** Ignore $\angle X(\omega)$. Instead, work with only $|X(\omega)|$, which is invariant to phase.

Outline

Magnitude DTFT is time-invariant, magnitude DFT is not

The magnitude DTFT is time-invariant. If $y[n] = x[n-n_0]$, then

$$|Y(\omega)| = \left|\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n}\right| = \left|e^{-j\omega n_0}\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right|$$

The magnitude DFT is NOT time-invariant:

$$|Y[k]| = \left|\sum_{n=0}^{N-1} x[n-n_0]e^{-j\frac{2\pi kn}{N}}\right| \neq \left|e^{-j\frac{2\pi kn_0}{N}}\sum_{\ell=0}^{N-1} x[\ell]e^{-j\frac{2\pi k\ell}{N}}\right|$$

By windowing the signal, we have selected N random samples. $|Y[k]|$ is therefore a random variable; it has the same distribution as $|X[k]|$, but not the same value.

Power Spectrum

Power spectrum is defined as the expected squared magnitude of the Fourier transform:

$$P_{XX}(\omega) = E\left[|X(\omega)|^2\right]$$
$$P_{XX}[k] = E\left[|X[k]|^2\right]$$

Finally! The power spectrum is a time-shift-invariant feature.

Power Spectral Estimators

Unfortunately, the power spectrum can't be computed: expectation ($E[\cdot]$) is the average over infinitely many instances of the random signal. There are three most common estimates:

- Periodogram (noisy but easy):

$$\widehat{P}_{XX}[k] = \left| \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \right|^2$$

- Bartlett's Method (average of the periodograms of multiple frames):

$$\widehat{P}_{XX}[k] = \frac{1}{T} \sum_{t=0}^{T-1} \left| \sum_{n=0}^{N-1} x[n - tN] e^{-j\frac{2\pi kn}{N}} \right|^2$$

- Welch's Method = average of the periodograms of windowed overlapping frames.

Outline

1. The Problem of Phase
2. Power Spectrum
3. **Autocorrelation**
4. Cross-correlation and cross-power spectrum
5. Filtering
6. Examples
7. Summary

Expectation is Linear

The expectation operator is linear. Remember that if $p_{X,Y}(x,y)$ is the joint pdf of random variables X and Y, then

$$E\left[aX+bY\right]=\int\int\left(ax+by\right)p_{X,Y}(x,y)dxdy$$
$$=aE\left[X\right]+b\left[Y\right]$$

Periodogram is DFT of a Correlation

The periodogram is the magnitude squared DFT, which are samples of the magnitude-squared DTFT:

$$\widehat{P}_{XX}(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega)$$

Multiplication and convolution are a Fourier transform pair:

$$X(\omega)X^*(\omega) \leftrightarrow x[n] * z[n],$$

where $z[n] = x^*[-n]$:

$$\begin{aligned} z[n] &= \frac{1}{2\pi}\int_0^{2\pi} X^*(\omega)e^{j\omega n} \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega(-n)}\right)^* \\ &= (x[-n])^* \end{aligned}$$

Power Spectrum ↔ Autocorrelation

Expectation is linear. Fourier tranform is linear. Convolution is the sum of products. So we can swap their order:

$$\begin{aligned}
P_{XX}(\omega) &= E\left[|X(\omega)|^2\right] \\
&= E\left[\text{DTFT}\left\{x[n] * x^*[-n]\right\}\right] \\
&= E\left[\text{DTFT}\left\{\sum_{n=-\infty}^{\infty} x[n]x^*[n-m]\right\}\right] \\
&= \text{DTFT}\left\{\sum_{n=-\infty}^{\infty} E\left[x[n]x^*[n-m]\right]\right\} \\
&= \text{DTFT}\left\{E\left[x[n]x^*[n-m]\right]\right\}
\end{aligned}$$

where the last line assumes that the signal is **wide-sense stationary** (WSS), meaning that its autocorrelation is independent of n.

The power spectrum is the DTFT of the autocorrelation

$$P_{XX}(\omega) \equiv E\left[|X(\omega)|^2\right]$$
$$R_{xx}[m] \equiv E\left[x[n]x^*[n-m]\right]$$
$$P_{XX}(\omega) = \text{DTFT}\left\{R_{XX}[m]\right\}$$

Properties of the autocorrelation

- **Conjugate symmetric:**

$$\begin{aligned} R_{XX}[-m] &= E\left[x[n]x^*[n+m]\right] \\ &= E\left[x[\ell-m]x^*[\ell]\right], \quad \ell = n+m \\ &= R_{XX}^*[m] \end{aligned}$$

  If $x[n]$ is real, then $R_{XX}[m]$ is real and symmetric.

- $R_{XX}[0]$ is the **signal power**:

$$\begin{aligned} R_{XX}[0] &= E\left[x[n]x^*[n]\right] \\ &= E\left[|x[n]|^2\right] \end{aligned}$$

Autocorrelation of a Sinusoid

As a first example, let's find the autocorrelation and power spectrum of a sinusoid. Suppose $x[n] = A\cos(\omega_0 n + \theta)$. Suppose ω_0 and A are known, and only θ is unknown (because we don't know how $n = 0$ is defined).

$$\begin{aligned} R_{XX}[m] &= E\left[x[n]x^*[n-m]\right] \\ &= A^2 E\left[\cos(\omega_0 n + \theta)\cos(\omega_0(n-m)+\theta)\right] \end{aligned}$$

Using the trig formula $\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$, we get

$$R_{XX}[m] = \frac{A^2}{2} E\left[\cos(\omega_0(2n-m)+2\theta) + \cos(\omega_0 m)\right]$$

But if θ is uniformly distributed, then $E\left[\cos\theta\right] = 0$, so

$$R_{XX}[m] = \frac{A^2}{2}\cos(\omega_0 m)$$

Notice that's real and symmetric!

Power Spectrum a Sinusoid

The power spectrum of $x[n] = A\cos(\omega_0 n + \theta)$ is

$$P_{XX}(\omega) = \text{DTFT}\left\{\frac{A^2}{2}\cos(\omega_0 m)\right\}$$
$$= \frac{\pi A^2}{2}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

Autocorrelation of White Noise

As a second example, let's find the autocorrelation of white noise. "White noise" is a type of signal whose power spectrum is a constant, independent of ω, like white light:

$$P_{XX}(\omega) = 1$$

The autocorrelation of such a signal is the inverse transform, which is a delta function:

$$\begin{aligned} R_{XX}[m] &= \delta[m] \\ &= \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \\ &= E\left[x[n]x^*[n-m]\right] \end{aligned}$$

In words: A white noise signal is one with uncorrelated samples.

Outline

Cross-Correlation

Suppose we have two signals, $x[n]$ and $y[n]$, that are both random, not equal to each other, but also not completely uncorrelated. We can write their cross-correlation as

$$R_{XY}[m] = E\left[x[n]y^*[n-m]\right]$$

The cross-power spectrum is

$$\begin{aligned} P_{XY}(\omega) &= \text{DTFT}\left\{R_{XY}[m]\right\} \\ &= E\left[\text{DTFT}\left\{x[n] * y^*[-n]\right\}\right] \\ &= E\left[X(\omega)Y^*(\omega)\right] \end{aligned}$$

Properties of the Cross-Correlation and Cross-power spectrum

- **Cross-correlation: conjugate antisymmetric in RVs and time:**

$$\begin{aligned} R_{XY}[m] &= E\left[x[n]y^*[n-m]\right] \\ &= \left(E\left[y[\ell]x^*[\ell+m]\right]\right)^*, \quad \ell = n-m \\ &= R^*_{YX}[-m] \end{aligned}$$

- **Cross-power spectrum: conjugate antisymmetric in RVs:**

$$\begin{aligned} P_{XY}(\omega) &= E\left[X(\omega)Y^*(\omega)\right] \\ &= P^*_{YX}(\omega) \end{aligned}$$

Cross-power spectra of two sinusoids at the same frequency

Suppose, first, that $x[n] = A\cos(\omega_0 n + \theta)$ and $y[n] = B\cos(\omega_0 n + \phi)$.

$$\begin{aligned} R_{XY}[m] &= AB \times E\left[\cos(\omega_0 n + \theta)\cos(\omega_0(n-m) + \phi)\right] \\ &= \frac{AB}{2}\cos(\omega_0 m + (\theta - \phi)) \end{aligned}$$

The cross-power spectrum is

$$P_{XY}(\omega) = \frac{AB\pi}{2}\left(e^{j(\theta-\phi)}\delta(\omega-\omega_0) + e^{-j(\theta-\phi)}\delta(\omega+\omega_0)\right)$$

Cross-power spectra of signal and shifted signal

Generalizing the previous slide: suppose $y[n] = ax[n - n_0] + v[n]$, where $v[n]$ is some noise signal that is uncorrelated with $x[n]$. Then

$$\begin{aligned} R_{XY}[m] &= E\left[x[n](ax^*[n - n_0 - m] + v^*[n - m])\right] \\ R_{XY}[m] &= aE\left[x[n]x^*[n - n_0 - m]\right] + E\left[x[n]v^*[n - m]\right] \\ &= aR_{XX}[m + n_0] + 0 \end{aligned}$$

The cross-power spectrum is

$$P_{XY}(\omega) = ae^{j\omega n_0} P_{XX}(\omega)$$

Outline

Real-world signals

- In the real world, pure tones and pure white-noise signals are rare.
- Most real-world signals can be modeled as $y[n] = h[n] * x[n]$, where $x[n]$ is a sum of pure tones and/or white noise, and $h[n]$ is a spectral shaping filter.
- Let's derive the cross-correlation, autocorrelation, cross-power spectrum, and power spectrum.

Cross-correlation

Suppose $y[n] = h[n] * x[n]$, then

$$\begin{aligned} R_{YX}[m] &= E\left[y[n]x^*[n-m]\right] \\ &= E\left[\left(\sum_{\ell=-\infty}^{\infty} h[\ell]x[n-\ell]\right) x^*[n-m]\right] \\ &= \sum_{\ell=-\infty}^{\infty} h[\ell]E\left[x[n-\ell]x^*[n-m]\right] \\ &= \sum_{\ell=-\infty}^{\infty} h[\ell]R_{XX}[m-\ell] \\ &= h[m] * R_{XX}[m] \end{aligned}$$

$$R_{XY}[m] = R^*_{YX}[-m] = h^*[-m] * R_{XX}[m]$$

Cross-power spectrum

Suppose $y[n] = h[n] * x[n]$, then

$$R_{XY}[m] = h^*[-m] * R_{XX}[m]$$
$$R_{YX}[m] = h[m] * R_{XX}[m]$$

Taking the DTFT, we get:

$$P_{XY}(\omega) = H^*(\omega)P_{XX}(\omega)$$
$$P_{YX}(\omega) = H(\omega)P_{XX}(\omega)$$

Autocorrelation

Suppose $y[n] = h[n] * x[n]$, then

$$\begin{aligned}
R_{YY}[m] &= E\left[y[n]y^*[n-m]\right] \\
&= E\left[\left(\sum_{\ell=-\infty}^{\infty} h[\ell]x[n-\ell]\right) y^*[n-m]\right] \\
&= \sum_{\ell=-\infty}^{\infty} h[\ell]E\left[x[n-\ell]y^*[n-m]\right] \\
&= \sum_{\ell=-\infty}^{\infty} h[\ell]R_{XY}[m-\ell] \\
&= h[m] * R_{XY}[m] \\
&= h[m] * h^*[-m] * R_{XX}[m]
\end{aligned}$$

The time-reversed convolution $(h[m] * h^*[-m])$ is conjugate antisymmetric.

Power Spectrum

Suppose $y[n] = h[n] * x[n]$, then

$$\begin{aligned} P_{YY}(\omega) &= \text{DTFT}\left\{R_{YY}[m]\right\} \\ &= \text{DTFT}\left\{h[m] * h^*[-m] * R_{XX}[m]\right\} \\ &= H(\omega)H^*(\omega)P_{XX}(\omega) \\ &= |H(\omega)|^2 P_{XX}(\omega) \end{aligned}$$

Outline

Examples

1. Suppose $x[n]$ is unit-variance white noise, and $y[n] = x[n] + ax[n-1]$. What is $P_{YY}(\omega)$?
2. Suppose $x[n]$ is unit-variance white noise, and $y[n] = x[n] - ay[n-1]$. What is $P_{YY}(\omega)$?

Outline

Summary

The power spectrum of white noise is constant; the power spectrum of a sinusoid is a cosine. If $y[n] = h[n] * x[n]$, then

$$R_{YX}[m] = h[m] * R_{XX}[m]$$
$$R_{XY}[m] = h^*[-m] * R_{XX}[m]$$
$$R_{YY}[m] = h[m] * h^*[-m] * R_{XX}[m]$$
$$P_{YX}(\omega) = H(\omega)P_{XX}(\omega)$$
$$P_{XY}(\omega) = H^*(\omega)P_{XX}(\omega)$$
$$P_{YY}(\omega) = |H(\omega)|^2 P_{XX}(\omega)$$