ECE 534 SP26 HW6 Solutions

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Problem 1. Define Doob's martingales and prove that they are indeed martingales. Use Doob's martingales to prove McDiarmid's inequality.

Solution.

PART 1: DEFINITION OF DOOB'S MARTINGALE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a random variable such that $\mathbb{E}[|X|] < \infty$. Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration, which is sequence of σ-algebras such that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}$. A **Doob's martingale** is a stochastic process $Z = (Z_n)_{n \geq 0}$ defined by

$$Z_n \coloneqq \mathbb{E}[X \mid \mathcal{F}_n]$$

for all $n \geq 0$.

PART 2: PROOF THAT A DOOB'S MARTINGALE IS INDEED A MARTINGALE

To show that the sequence $(Z_n)_{n \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$, we must verify three properties for all $n \geq 0$:

1) **Adaptability:** Z_n must be $\mathcal{F}_n$-measurable. By the definition of conditional expectation, $\mathbb{E}[X \mid \mathcal{F}_n]$ is inherently $\mathcal{F}_n$-measurable.
2) **Integrability:** We need to show that $\mathbb{E}[|Z_n|] < \infty$. We can decompose X into its positive and negative parts, $X = X^+ - X^-$, where both X^+ and X^- are non-negative random variables. Since X is integrable by assumption, we know $\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-] < \infty$.
   By the linearity of conditional expectation, we have
   $$Z_n = \mathbb{E}[X^+ \mid \mathcal{F}_n] - \mathbb{E}[X^- \mid \mathcal{F}_n].$$
   Using the triangle inequality, we obtain
   $$|Z_n| \leq |\mathbb{E}[X^+ \mid \mathcal{F}_n]| + |\mathbb{E}[X^- \mid \mathcal{F}_n]|.$$
   Because X^+ and X^- are non-negative, the inequality above becomes
   $$|Z_n| \leq \mathbb{E}[X^+ \mid \mathcal{F}_n] + \mathbb{E}[X^- \mid \mathcal{F}_n] = \mathbb{E}[|X| \mid \mathcal{F}_n].$$
   Taking the expectation on both sides and applying the tower property, we get
   $$\mathbb{E}[|Z_n|] \leq \mathbb{E}[\mathbb{E}[|X| \mid \mathcal{F}_n]] = \mathbb{E}[|X|] < \infty.$$
   Thus, Z_n is integrable for all n.
3) **Martingale Property:** We must show that for any $n \geq 1$, $\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] = Z_{n-1}$ almost surely. Using the definition of Z_n and the tower property of conditional expectation (since $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$), we obtain:
   $$\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_n] \mid \mathcal{F}_{n-1}] = \mathbb{E}[X \mid \mathcal{F}_{n-1}] = Z_{n-1}.$$

Since all three properties hold for all $n \geq 0$, the process $(Z_n)_{n \geq 0}$ is indeed a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$.

PART 3: PROOF OF MCDIARMID'S INEQUALITY USING DOOB'S MARTINGALE

Theorem 1.1 (McDiarmid's Inequality)**.** *[1, Proposition 10.11] Let* $X_1, \dots, X_n$ *be independent random variables taking values in sets* $\mathcal{X}_1, \dots, \mathcal{X}_n$ *respectively. Let* $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ *be a function that satisfies the following* bounded difference property*: There exist* $c_1, \dots, c_n > 0$ *such that all* $i \in \{1, \dots, n\}$ *and for all* $x_1 \in \mathcal{X}_1, \dots, x_n \in \mathcal{X}_n$ *and* $x_i' \in \mathcal{X}_i$*, we have*

$$|f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)| \le c_i.$$

Then, for any $t > 0$*, we have*

$$\mathbb{P}(f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \ge t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right), \tag{1.1}$$

$$\mathbb{P}(f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \le -t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right), \tag{1.2}$$

and consequently,

$$\mathbb{P}(|f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)]| \ge t) \le 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right). \tag{1.3}$$

Before proceeding, we state the key inequality that governs the concentration of martingales with bounded differences.

Lemma 1.1 (The Azuma-Hoeffding Inequality)**.** *Let* $(Z_i)_{i=0}^n$ *be a martingale with respect to a filtration* $(\mathcal{F}_i)_{i=0}^n$*. Assume that there exists random variables* $A_1, \dots, A_n$*,* $B_1, \dots, B_n$*, where* A_i *and* B_i *are* $\mathcal{F}_{i-1}$*-measurable for each* $1 \le i \le n$*, and positive constants* $c_1, \dots, c_n$*, such that both the following hold for each* $1 \le i \le n$*:*

- $B_i - A_i \le c_i$ *almost surely.*
- $D_i := Z_i - Z_{i-1}$ *satisfies* $A_i \le D_i \le B_i$ *almost surely.*

Then, for any $t > 0$*, we have*

$$\mathbb{P}(Z_n - Z_0 \ge t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right).$$

Remark. The proof of Lemma 1.1 is essentially using the tower property of conditional expectation and repeatedly applying Hoeffding's Lemma [1, Lemma 10.8]. See https://en.wikipedia.org/wiki/Azuma%27s_inequality#A_general_form_of_Azuma's_inequality for more details. A variant of Lemma 1.1 can be seen in [1, Proposition 10.9].

Proof of Theorem 1.1. Note that it suffices to prove (1.1), since applying (1.1) to $-f$ yields (1.2), and using the union bound to combine (1.1) and (1.2) immediately leads to (1.3).

We seek to construct a Doob's martingale. Before that, we need to check that $f(X_1, \dots, X_n)$ is integrable. Pick arbitrary $x_1' \in \mathcal{X}_1, x_2' \in \mathcal{X}_2, \dots, x_n' \in \mathcal{X}_n$. By the bounded difference property of f and the triangle inequality, we have

$$\begin{aligned}
|f(X_1, \dots, X_n)| &= |f(X_1, \dots, X_n) - f(x_1', \dots, x_n') + f(x_1', \dots, x_n')| \\
&\le |f(X_1, \dots, X_n) - f(x_1', \dots, x_n')| + |f(x_1', \dots, x_n')| \\
&\le \sum_{i=1}^n c_i + |f(x_1', \dots, x_n')|.
\end{aligned}$$

Taking expectation shows that $\mathbb{E}\left[|f(X_1, \dots, X_n)|\right] < \infty$.

Now we can construct the Doob's martingale. Let $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ be the filtration generated by the first i variables, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (the trivial σ-algebra). Define the Doob's martingale:

$$Z_i := \mathbb{E}[f(X_1, \dots, X_n) \mid \mathcal{F}_i].$$

Note that $Z_0 = \mathbb{E}[f(X_1, \dots, X_n)]$ and $Z_n = f(X_1, \dots, X_n)$. The total deviation from the mean is exactly $Z_n - Z_0$.

To apply the Azuma-Hoeffding inequality, we must bound the conditional range of the martingale differences $D_i = Z_i - Z_{i-1}$. Define two random variables U_i and L_i which depend only on $X_1, \dots, X_{i-1}$ (and are therefore $\mathcal{F}_{i-1}$-measurable):

$$\begin{aligned} U_i &\coloneqq \sup_{x \in \mathcal{X}_i} \mathbb{E}[f(X_1, \dots, X_{i-1}, x, X_{i+1}, \dots, X_n) \mid \mathcal{F}_{i-1}], \\ L_i &\coloneqq \inf_{x \in \mathcal{X}_i} \mathbb{E}[f(X_1, \dots, X_{i-1}, x, X_{i+1}, \dots, X_n) \mid \mathcal{F}_{i-1}]. \end{aligned}$$

Now we claim that $L_i \le Z_i \le U_i$ for each i. Define the deterministic function

$$g_i(x_1, \dots, x_i) \coloneqq \mathbb{E}[f(x_1, \dots, x_i, X_{i+1}, \dots, X_n)].$$

Because $X_1, \dots, X_n$ are independent, conditioning on $\mathcal{F}_i$ simply integrates out the unobserved variables. Thus, Z_i is just this deterministic function evaluated at the random observations:

$$Z_i = g_i(X_1, \dots, X_i).$$

Similarly, we have

$$\begin{aligned} U_i &= \sup_{x \in \mathcal{X}_i} g_i(X_1, \dots, X_{i-1}, x) \\ L_i &= \inf_{x \in \mathcal{X}_i} g_i(X_1, \dots, X_{i-1}, x) \end{aligned}$$

Since the actual realization of X_i must belong to $\mathcal{X}_i$, the value Z_i is naturally bounded by its infimum and supremum over that coordinate, proving $L_i \le Z_i \le U_i$.

Subtracting Z_{i-1} from all sides, we find that the difference D_i is almost surely bounded by

$$L_i - Z_{i-1} \le D_i \le U_i - Z_{i-1}. \tag{1.4}$$

The length of this interval is exactly $U_i - L_i$. By the bounded difference property of the function f, varying the i-th coordinate can change the expected value by at most c_i. Therefore, we have

$$U_i - L_i \le c_i. \tag{1.5}$$

From (1.4) and (1.5), we can directly apply Lemma 1.1 to get

$$\mathbb{P}(Z_n - Z_0 \ge t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right)$$

Substituting $Z_n = f(X_1, \dots, X_n)$ and $Z_0 = \mathbb{E}[f(X_1, \dots, X_n)]$ yields the desired result:

$$\mathbb{P}(f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \ge t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right).$$

This completes the proof. □

Problem 2. [1, Problem 4.7].

Solution. See the solution set for HW5.

Problem 3. [1, Problem 4.9].

Solution. 4.9 Brownian bridge

(a) See Figure 1.

(b)

For $s, t \in [0, 1]$, if $s \le t$:

$$R_B(s, t) = E[(W_s - sW_1)(W_t - tW_1)] = s - st - st + st = s - st = s(1 - t).$$

It follows by switching s and t that if $s \ge t$: $R_B(s, t) = (1 - s)t$. Therefore,

$$R_B(s, t) = \begin{cases} s(1 - t) & \text{for } 0 \le s \le t \le 1 \\ t(1 - s) & \text{for } 0 \le t \le s \le 1 \end{cases}$$

Equivalently, $R_B(s, t) = (s \wedge t)(1 - s \vee t)$ for $s, t \in [0, 1]$.

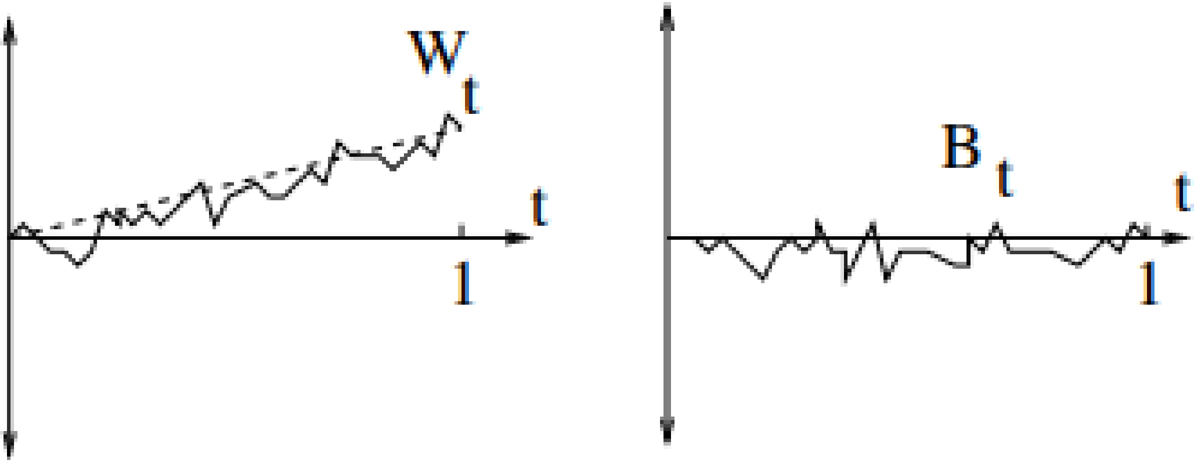

Fig. 1: Corresponding sample paths of a Brownian motion and a Brownian bridge

(c)
Letting $\rho(s,t)$ denote the correlation coefficient between X_s and X_t, we find that for $s < t$:

$$\rho(s,t) = \frac{\mathrm{Cov}(B_s, B_t)}{\sqrt{\mathrm{Var}(B_s)\mathrm{Var}(B_t)}} = \frac{s(1-t)}{\sqrt{s(1-s)\cdot t(1-t)}} = \sqrt{\frac{s}{1-s}}\sqrt{\frac{1-t}{t}}.$$

Therefore, for $0 \leq r < s < t \leq 1$, $\rho(r,s)\rho(s,t) = \rho(r,t)$, so that by Example 4.8.3 of the notes, the Gaussian process B is a Markov process.

(d)
It is enough to show that $\mathrm{Cov}(B_t, W_1) = 0$ for $0 \leq t \leq 1$, because uncorrelated jointly Gaussian vectors are independent. But

$$\mathrm{Cov}(B_t, W_1) = \mathrm{Cov}(W_t - tW_1, W_1) = \mathrm{Cov}(W_t, W_1) - t\,\mathrm{Cov}(W_1, W_1) = t - t = 0.$$

(e)
For $0 \leq s \leq t < 1$, $\frac{s}{1-s} \leq \frac{t}{1-t}$. Therefore,

$$\begin{aligned} R_X(s,t) &= \mathrm{Cov}\left((1-s)W_{\frac{s}{1-s}}, (1-t)W_{\frac{t}{1-t}}\right) \\ &= (1-s)(1-t)\,\mathrm{Cov}\left(W_{\frac{s}{1-s}}, W_{\frac{t}{1-t}}\right) \\ &= (1-s)(1-t)\frac{s}{1-s} = s(1-t). \end{aligned}$$

This expression holds for $0 \leq s \leq t = 1$ as well. Similarly, if $0 \leq t \leq s \leq 1$, $R_X(s,t) = (1-s)t$. That is, X has the same autocorrelation function as the process B. Since both X and B are mean zero Gaussian random processes with identical autocorrelation functions, they have the same finite-dimensional distributions. Also, both have continuous sample paths. So X is also a Brownian bridge process.

Problem 4. [1, Problem 4.10].

Solution. See [1, Chapter 12].

Problem 5. [1, Problem 4.11].

Solution. 4.11 Some Poisson process calculations

(a) (Method 1) Given $N_2 = 2$, the distribution of the locations of the first two jump points are as if they are independently and uniformly distributed over the interval $[0,2]$. Thus, each falls in the second half of the interval with probability $1/2$, and they both fall into the second interval with probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Thus, at least one of them falls in the first half of the interval with probability $1 - \frac{1}{4} = \frac{3}{4}$.

(Method 2) Since N_1 and $N_2 - N_1$ are independent, Poi(λ) random variables,

$$\begin{aligned} P\{N_1 \geq 1, N_2 = 2\} &= P\{N_1 = 1, N_2 = 2\} + P\{N_1 = 2, N_2 = 2\} \\ &= P\{N_1 = 1, N_2 - N_1 = 1\} + P\{N_1 = 2, N_2 - N_1 = 0\} \\ &= (\lambda e^{-\lambda})(\lambda e^{-\lambda}) + \left(\frac{\lambda^2 e^{-\lambda}}{2}\right) e^{-\lambda} = \frac{3\lambda^2 e^{-2\lambda}}{2} \end{aligned}$$

and $P\{N_2\} = \frac{(2\lambda)^2 e^{-2\lambda}}{2}$. Thus, $P(N_1 \geq 1 | N_2 = 2) = \frac{3\lambda^2 e^{-2\lambda}}{2} / \left(\frac{(2\lambda)^2 e^{-2\lambda}}{2}\right) = \frac{3}{4}$.

(b) $P\{N_1 \geq 1\} = 1 - e^{-\lambda}$. Therefore, $P(N_2 \geq 2 | N_1 \geq 1) = \frac{3\lambda^2 e^{-\lambda}}{2} / (1 - e^{-\lambda})$.

(c) Yes. The process N is Markov (because N_0 is constant and N has independent increments), and since X_t and N_t are functions of each other for each t, the process X is also Markov. The state space for X is $S = \{0, 1, 4, 9, \ldots\}$. For $i, j \in \mathbb{Z}$ and $t, \tau \geq 0$,

$$p_{i^2, j^2}(\tau) = P(X_{t+\tau} = j^2 | X_t = i^2) = \begin{cases} \dfrac{(\lambda\tau)^{j-i} e^{-\lambda\tau}}{(j-i)!} & 0 \leq i \leq j \\ 0 & \text{else} \end{cases}$$

Problem 6. [1, Problem 4.22].

Solution. See [1, Chapter 12].

REFERENCES

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf