ECE 534 SP26 HW5 Solutions

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Solution to [1, Problem 2.31]

Since $\mathbb{E}\left[X_1^2\right] = 2 > 1$, Cramér's theorem [1, Theorem 2.19] implies that Chernoff bound gives the exact exponent. That is, we have $b = \ell(2)$ (see the notations in the cited theorem), which we compute. For $a > 0$, we have

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(\frac{1}{2a})}} dx = \sqrt{\frac{\pi}{a}},$$

so the log moment generating function of X_1^2 is given by

$$M(\theta) = \ln \mathbb{E}\left[e^{\theta X_1^2}\right] = \ln \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2(\frac{1}{2}-\theta)} dx = -\frac{1}{2}\ln(1-2\theta).$$

It follows that

$$\ell(a) = \max_{\theta} \left\{\theta a + \frac{1}{2}\ln(1-2\theta)\right\}.$$

By differentiation, we found that

$$\begin{aligned} \theta^* &= \operatorname*{argmax}_{\theta} \left\{\theta a + \frac{1}{2}\ln(1-2\theta)\right\} \\ &= \frac{1}{2}\left(1 - \frac{1}{a}\right), \end{aligned}$$

which gives

$$\ell(a) = \frac{1}{2}\left(a - \ln a - 1\right).$$

The exponent of interest b is then

$$b = \ell(2) = \frac{1}{2}(1 - \ln 2) = 0.1534.$$

The numerical value of interest is

$$e^{-100b} \approx 2.18 \times 10^{-7}.$$

Solution to [1, Problem 4.3]

Since $X_t = A\cos(2\pi Vt)\cos(\Theta) - A\sin(2\pi Vt)\sin(\Theta)$ and $\mathbb{E}[\cos(\Theta)] = \mathbb{E}[\sin(\Theta)] = 0$, it follows that $\mu_X(t) \equiv 0$. The autocorrelation function of X is given by

$$\begin{aligned} R_X(s,t) &= \mathbb{E}\left[A^2\right]\mathbb{E}\left[\cos(2\pi Vs + \Theta)\cos(2\pi Vt + \Theta)\right] \\ &= \frac{\mathbb{E}\left[A^2\right]}{2}\mathbb{E}\left[\cos(2\pi V(s-t)) + \cos(2\pi V(s+t) + 2\Theta)\right] \\ &= \frac{\mathbb{E}\left[A^2\right]}{2}\left(\int_0^5 \frac{1}{5}\cos(2\pi v(s-t))dv + 0\right) \\ &= 4\mathrm{sinc}(10(s-t)), \end{aligned}$$

where we used the fact that $\mathbb{E}\left[A^2\right] = (\mathbb{E}[A])^2 + \mathrm{Var}(A) = 8$ and the definition

$$\mathrm{sinc}(u) := \begin{cases} \frac{\sin(\pi u)}{\pi u}, & \text{if } u \neq 0, \\ 1, & \text{if } u = 0. \end{cases}$$

Yes, the random process X is wide-sense stationary (WSS). By definition, a random process is WSS if it satisfies two conditions:

1) Its mean function is constant for all time t.
2) Its autocorrelation function $R_X(s,t)$ depends only on the time difference $\tau = s - t$.

As derived above, the mean function is $\mu_X(t) = 0$, which is constant. Furthermore, the autocorrelation function $R_X(s,t) = 4\text{sinc}(10(s-t))$ depends entirely on the difference $s-t$. Since both conditions are satisfied, X is WSS.

Solution to [1, Problem 4.7]

(a) The event in question is the same as $\{2W_3 \geq W_2 + W_4 + 2\}$ or $\{(W_3 - W_2) - (W_4 - W_3) \geq 2\}$. Since $W_3 - W_2$ and $W_4 - W_3$ are mutually independent $N(0,1)$ random variables, their difference is a $N(0,2)$ random variable. Thus, the probability in question is $Q(\frac{2}{\sqrt{2}})$ or $Q(\sqrt{2})$.

ALTERNATIVELY, the event in question is the same as $\{Z \geq 2\}$ where $Z = -W_2 + 2W_3 - W_4$. Since W is a Gaussian process, Z is a Gaussian random variable. Since the W's have mean zero, $\mathbb{E}[Z] = 0$. Finally, $\text{Var}(Z) = \text{Cov}(Z,Z) = \text{Var}(W_2) + 4\text{Var}(W_3) + \text{Var}(W_4) - 4\text{Cov}(W_2,W_3) - 4\text{Cov}(W_3,W_4) + 2\text{Cov}(W_2,W_4) = 2 + 12 + 4 - 8 - 12 + 4 = 2$. Thus, the probability in question is $\mathbb{P}(Z \geq 2) = Q(\frac{2}{\sqrt{2}})$ or $Q(\sqrt{2})$.

(b) For any t, $\frac{W_t}{\sqrt{t}}$ has the $N(0,1)$ distribution so for any t, so $\frac{W_t^2}{t}$ has the same distribution as Z^2, where Z is a $N(0,1)$ random variable. It follows that the limiting distribution of $\frac{W_t^2}{t}$ is also the same as Z^2. The PDF is found as follows. For $c > 0$, we have $\mathbb{P}\left(Z^2 > c\right) = 2\mathbb{P}(Z > \sqrt{c}) = 2Q(\sqrt{c})$. Taking the derivative with respect to c and using the chain rule and definition of Q yields for $c > 0$ that

$$f_{Z^2}(c) = -\frac{d\mathbb{P}\left(Z^2 > c\right)}{dc} = 2\frac{1}{2\sqrt{c}}\frac{1}{\sqrt{2\pi}}e^{-c/2} = \frac{1}{\sqrt{2\pi c}}e^{-c/2}.$$

At the same time, it is clear that for $c \leq 0$ we have $\mathbb{P}\left(Z^2 > c\right) = 1$, and thus $f_{Z^2}(c) = 0$ for $c \leq 0$. In summary, the PDF of Z^2, which is the PDF of the limiting distribution of interest, writes

$$f_{Z^2}(c) = \begin{cases} \frac{1}{\sqrt{2\pi c}}e^{-c/2}, & \text{if } c > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. Z^2 has the chi-square distribution with one degree of freedom, or, equivalently, the gamma distribution with parameter $1/2$.

The solution to [1, Problem 2.33] can be seen in the solution set for HW4, and the solutions to the other two problems can be found in [1, Chapter 12].

References

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf