ECE 534 SP26 HW3 Solutions

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Problem 1. Let $f(x)$ be a function with domain $[0,1]$ such that $f(x)=1$ if x is a rational number, and $f(x)=0$ otherwise. What is the Riemann integral of this function? Define the Lebesgue integral as we did in class and compute the Lebesgue integral of the function.

Solution.

The Riemann Integral. We first determine whether $f(x)$ is Riemann integral or not. To evaluate its Riemann integrability, we consider the upper and lower Darboux sums (also called the upper and lower Riemann sums) for an arbitrary partition $P=\{x_0,x_1,\dots,x_n\}$ of $[0,1]$. Because the rational numbers are dense in $\mathbb{R}$, every subinterval $[x_{i-1},x_i]$ contains at least one rational number. Therefore, the supremum of $f(x)$ on any subinterval $[x_{i-1},x_i]$ is $M_i=1$. The upper Darboux sum is therefore

$$U(f,P)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})=\sum_{i=1}^{n} 1\cdot(x_i-x_{i-1})=1.$$

Similarly, because the irrational numbers are dense in $\mathbb{R}$, every subinterval $[x_{i-1},x_i]$ contains at least one irrational number. Thus, the infimum of $f(x)$ on any subinterval is $m_i=0$. The lower Darboux sum is then

$$L(f,P)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})=\sum_{i=1}^{n} 0\cdot(x_i-x_{i-1})=0.$$

Since the upper Riemann integral and lower Riemann integral are strictly not equal ($\inf U(f,P)=1\neq 0=\sup L(f,P)$), the Riemann integral of $f(x)$ does not exist.

Definition of the Lebesgue Integral. We now recall the definition Lebesgue integral. Let λ denote the Lebesgue measure. For a non-negative simple function $\phi(x)=\sum_{i=1}^{k} a_i \mathbb{1}_{A_i}(x)$, where A_i are disjoint measurable sets, $\mathbb{1}$ represents the indicator function, and a_i are positive real numbers, the Lebesgue integral of ϕ is defined as:

$$\int \phi\, d\lambda=\sum_{i=1}^{k} a_i\lambda(A_i).$$

For a general non-negative measurable function f, the Lebesgue integral is defined as the supremum of the integrals of all simple functions bounded above by f:

$$\int f\, d\lambda=\sup\left\{\int \phi\, d\lambda : 0\le\phi\le f,\ \phi \text{ is simple}\right\}.$$

Lastly, for a general measurable function f that takes on both positive and negative values, we decompose it into its positive and negative parts, defined respectively as:

$$f^+(x)=\max\{f(x),0\} \quad \text{and} \quad f^-(x)=\max\{-f(x),0\}.$$

Note that both f^+ and f^- are non-negative measurable functions. We can express the original function and its absolute value in terms of these two non-negative parts:

$$f(x)=f^+(x)-f^-(x) \quad \text{and} \quad |f(x)|=f^+(x)+f^-(x).$$

Using the definition of the integral for non-negative functions established above, the Lebesgue integral of f is defined as the difference:

$$\int f\, d\lambda=\int f^+\, d\lambda-\int f^-\, d\lambda,$$

provided that at least one of the integrals $\int f^+\, d\lambda$ or $\int f^-\, d\lambda$ is finite (to avoid the undefined expression $\infty-\infty$).

We say that f is *Lebesgue integrable* if both of these integrals are finite. Equivalently, f is Lebesgue integrable if and only if the integral of its absolute value is finite:

$$\int |f| \, d\lambda < \infty.$$

Computation of the Lebesgue Integral. We now calculate the Lebesgue integral of $f(x)$ given in the problem description. Note that $f(x)$ is exactly the indicator function of the rational numbers within $[0,1]$. That is, we can express this $f(x)$ as $f(x) = \mathbb{1}_A(x)$, where the set $A := \mathbb{Q} \cap [0,1]$. Applying the definition of the Lebesgue integral for simple functions yields

$$\int_{[0,1]} f(x) \, d\lambda = \lambda(A).$$

The set of rational numbers $\mathbb{Q}$ is countably infinite, and thus so is the set A. A fundamental property of the Lebesgue measure is that the measure of any countable set of points in $\mathbb{R}$ is exactly zero. Therefore, $\lambda(A) = 0$.

Substituting this back into our integral yields

$$\int_{[0,1]} f(x) \, d\lambda = 0.$$

Thus, while the Riemann integral does not exist, the Lebesgue integral of the function is exactly 0.

Problem 2. Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of $\mathcal{N}(0,1)$ (standard Gaussian) random variables. Define the events

$$A_n := \{|X_n| > \sqrt{2 \log n}\}.$$

Determine whether the events A_n occur infinitely often. Hint: Use Borel-Cantelli's lemma.

Solution. Define the function

$$Q(c) := \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} e^{-\frac{x^2}{2}} \, dx.$$

That is, $Q(c)$ is exactly $\mathbb{P}(Z > c)$ for $Z \sim \mathcal{N}(0,1)$. It follows that by symmetry we can write

$$\mathbb{P}(A_n) = 2Q(\sqrt{2 \log n}). \tag{2.1}$$

We now derive a lower bound on $Q(c)$ for any $c > 0$. We first write

$$\begin{aligned} Q(c) &= \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} \frac{x}{x} e^{-\frac{x^2}{2}} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} u dv, \end{aligned}$$

where $u := \frac{1}{x}$ and $dv := x e^{-\frac{x^2}{2}} dx$. Using integration by parts (i.e. $\int u dv = uv - \int v du$), we get

$$\begin{aligned} Q(c) &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-\frac{x^2}{2}}}{x} \Bigg|_{x=c}^{\infty} - \int_{x=c}^{\infty} \frac{e^{-\frac{x^2}{2}}}{x^2} dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{c^2}{2}}}{c} - \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} \frac{e^{-\frac{x^2}{2}}}{x^2} dx, \end{aligned} \tag{2.2}$$

where we used the fact that $\lim_{x \to \infty} \frac{e^{-\frac{x^2}{2}}}{x} = 0$. To get a lower bound on $Q(c)$, it suffices to get an upper bound on $\frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} \frac{e^{-\frac{x^2}{2}}}{x^2} dx$, which can be simply obtained as

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} \frac{e^{-\frac{x^2}{2}}}{x^2} dx &\leq \frac{1}{\sqrt{2\pi}} \int_{x=c}^{\infty} \frac{e^{-\frac{x^2}{2}}}{c^2} dx \\ &= \frac{1}{c^2} Q(c). \end{aligned} \tag{2.3}$$

Putting (2.3) into (2.2) yields

$$Q(c) \geq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{c^2}{2}}}{c} - \frac{1}{c^2} Q(c),$$

and thus by rearranging the terms, we get

$$Q(c) \geq \frac{c}{1+c^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}. \tag{2.4}$$

Applying the bound in (2.4) with $c = \sqrt{2 \log n}$, we get

$$\begin{aligned} Q(\sqrt{2\log n}) &\geq \frac{\sqrt{2\log n}}{1+2\log n} \frac{1}{\sqrt{2\pi}} e^{-\frac{2\log n}{2}} \\ &= \frac{\sqrt{2\log n}}{1+2\log n} \frac{1}{\sqrt{2\pi}} \frac{1}{n} \\ &\geq C \frac{1}{n\sqrt{\log n}} \end{aligned} \tag{2.5}$$

for some constant C and for n large enough (say for all $n \geq N$). Then, putting (2.5) into (2.1) yields for $n \geq N$ that

$$\mathbb{P}(A_n) \geq 2C \frac{1}{n\sqrt{\log n}}. \tag{2.6}$$

From (2.6), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(A_n) &\geq \sum_{n=N}^{\infty} 2C \frac{1}{n\sqrt{\log n}} \\ &= \infty. \end{aligned} \tag{2.7}$$

(The divergence of the series could be checked by the integral test.) Finally, since $(A_n)_{n\geq 1}$ are independent events, by the second Borel-Cantelli lemma, we can conclude from (2.7) that $\mathbb{P}(\limsup_{n\to\infty} A_n) = 1$. That is, with probability one, the events A_n happen infinitely often.

Remark 2.1. Actually we have for all $c > 0$ that

$$\frac{c}{1+c^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} \leq Q(c) \leq \frac{1}{c} \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}. \tag{2.8}$$

In the previous version of this solution set, the preparer made a mistake and used the upper bound on $Q(c)$ in (2.8) instead. The main takeaway is that if c goes to infinity (like $c = \sqrt{2\log n}$ in this problem), then the upper bound and the lower bound in (2.8) behave asymptotically the same. It may not be necessary that you remember the exact formula in (2.8), but the key point is that their derivations are similar: You first turn $e^{-\frac{x^2}{2}}$ into $xe^{-\frac{x^2}{2}}$ by multiplying $\frac{x}{x}$, and then you can either directly bound the integral or use integration by parts. See the following Wikipedia page for more details regarding the Q-function: https://en.wikipedia.org/wiki/Q-function#Bounds_and_approximations.

Problem 3. Let $(U_n)_{n\geq 1}$ be independent Unif$[0,1]$ random variables. Define a sequence of random variables $(X_n)_{n\geq 1}$ by

$$X_n(\omega) := n\mathbb{1}_{|U_n(\omega)-\omega|\leq 1/n},$$

where $\omega \in [0,1]$ and $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function. Determine if the sequence $(X_n)_{n\geq 1}$ converges almost surely. Determine if it converges in probability.

Solution. To make the expression $U_n(\omega)$ make sense in the first place, we assume that $(U_n)_{n\geq 1}$ is defined on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$. We provide the solution to this problem with the **additional assumption** that $(U_n)_{n\geq 1}$ are mutually independent with the random variable

$$U(\omega) = \omega,$$

which is defined on the same probability space and follows the distribution Unif$[0,1]$ as well. Note that we can now simply write $X_n = n\mathbb{1}_{|U_n - U|\leq 1/n}$.

Convergence in probability. First, we claim that $X_n \xrightarrow{\mathbb{P}} 0$. By definition, we need to show whether $\lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = 0$ for any given $\varepsilon > 0$.

Since X_n only takes the values 0 and n, for any chosen $0 < \varepsilon < 1$ and for all $n \geq 1$, we have:

$$\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(X_n = n) = \mathbb{P}\left(|U_n - U| \leq \frac{1}{n}\right).$$

Because U_n and U are independent standard uniform random variables, their joint probability density function is $f_{U_n,U}(x, y) = 1$ over the unit square $[0, 1]^2$. The probability is simply the geometric area of the region where $|x - y| \leq 1/n$.

This region is a diagonal band across the unit square. We can find its area by subtracting the area of the two large empty corner triangles from the total area of the square. Each triangle has a base and height of $(1 - 1/n)$:

$$\mathbb{P}(X_n = n) = 1 - \left(1 - \frac{1}{n}\right)^2 = \frac{2}{n} - \frac{1}{n^2}.$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = \lim_{n\to\infty} \left(\frac{2}{n} - \frac{1}{n^2}\right) = 0.$$

In conclusion, the sequence $(X_n)_{n\geq1}$ converges to 0 in probability, which completes the proof of the claim.

Almost sure convergence. Since almost sure convergence implies convergence in probability to the same limit (see [1, Proposition 2.7 (a)]), all we need to check is whether X_n converges to 0 a.s. or not. That is, we need to determine if $\mathbb{P}(\lim_{n\to\infty} X_n = 0) = 1$. Because X_n alternates between 0 and n, the sequence fails to converge to 0 if the event $A_n = \{|U_n - U| \leq 1/n\}$ happens infinitely often (i.o.).

To analyze this, we condition on $U = u$. Given a fixed value $U = u$, the events $A_n(u) = \{|U_n - u| \leq 1/n\}$ are mutually independent because the underlying sequence $(U_n)_{n\geq1}$ is mutually independent.

For any fixed $u \in (0, 1)$, there exists an integer N large enough such that for all $n \geq N$, the interval $[u - 1/n, u + 1/n]$ is entirely contained within $[0, 1]$. For all such n, the conditional probability is:

$$\mathbb{P}(A_n \mid U = u) = \frac{2}{n}.$$

Now we sum these conditional probabilities:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n \mid U = u) \geq \sum_{n=N}^{\infty} \frac{2}{n} = \infty.$$

Because the conditionally evaluated events are independent and the sum of their probabilities diverges, the second Borel-Cantelli lemma guarantees that:

$$\mathbb{P}(A_n \text{ i.o.} \mid U = u) = 1.$$

This holds for all $u \in (0, 1)$. Using the continuous version of the Law of Total Probability, we integrate out U to find the unconditional probability:

$$\mathbb{P}(A_n \text{ i.o.}) = \int_0^1 \mathbb{P}(A_n \text{ i.o.} \mid U = u) f_U(u)\, du = \int_0^1 (1)(1)\, du = 1.$$

Since the probability of the sequence $(X_n)_{n\geq1}$ jumping to n infinitely often is 1, the sequence almost surely does not converge to 0. It follows that X_n does not converge a.s.

Problem 4. Let $(X_n)_{n\geq1}$ be a sequence of random variables defined over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let X be another random variable. Suppose that

$$X_n \xrightarrow{\mathbb{P}} X, \text{ i.e.}, \forall \varepsilon > 0 : \mathbb{P}(|X_n - X| > \varepsilon) \to 0.$$

Assume in addition that for every $\varepsilon > 0$, the probabilities of deviations satisfy the summability condition

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty.$$

Prove that in this case the sequence also converges in the almost sure sense. Provide an intuition for this result.

Solution. Let $\varepsilon > 0$ be given. Define the sequence of events $A_n = \{|X_n - X| > \varepsilon\}$.

The problem statement provides the summability condition:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty.$$

By the First Borel-Cantelli Lemma, this finite sum implies that the probability of the events A_n occurring infinitely often (i.o.) is zero:

$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0.$$

This means that with probability 1, the events A_n occur only finitely many times. Equivalently, for almost every outcome $\omega \in \Omega$, there exists an integer $N(\omega)$ such that for all $n \geq N(\omega)$, we have $|X_n(\omega) - X(\omega)| \leq \varepsilon$.

To show almost sure convergence rigorously, this must hold simultaneously for all $\varepsilon > 0$. Let us choose a sequence $\varepsilon_k := 1/k$ for $k \geq 1$. For each k, we then let $E_k := \{\limsup_{n\to\infty} |X_n - X| > 1/k\}$. From our application of Borel-Cantelli above, $\mathbb{P}(E_k) = 0$ for all $k \geq 1$.

The event that X_n does not converge to X can be written as the union of these events:

$$\left\{\lim_{n\to\infty} X_n \neq X\right\} = \bigcup_{k=1}^{\infty} E_k.$$

Using the union bound, we get:

$$\mathbb{P}\left(\lim_{n\to\infty} X_n \neq X\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mathbb{P}(E_k) = \sum_{k=1}^{\infty} 0 = 0.$$

Therefore, $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$, which by definition means that X_n converges to X almost surely.

Intuition. Convergence in probability merely requires the "size" of the bad event $\mathbb{P}(|X_n - X| > \varepsilon)$ to shrink towards zero as $n \to \infty$. It allows for a specific outcome ω to occasionally result in a large deviation, as long as at any single step n, the total likelihood of such deviations is small.

However, the summability condition $\sum \mathbb{P}(|X_n - X| > \varepsilon) < \infty$ is much stronger. It implies that the *expected total number of times* the sequence deviates from X by more than ε is finite. If an event is expected to happen only a finite number of times, any given outcome ω will eventually stop seeing these large deviations. Once you go far enough down the sequence, the random variable X_n gets "trapped" within the ε-tube around X forever, which is exactly what is required for point-wise (almost sure) convergence.

Problem 5. Prof. Hajek's text [1], 2.4, 2.9, 2.10.

Solution. For [1, Problem 2.4] and [1, Problem 2.10], please refer to [1, Chapter 12].

Here is the solution to [1, Problem 2.9]:

(a) The sequence $X_n(\omega)$ is monotone nondecreasing in n for each ω. Also, by induction on n, $0 \leq X_n(\omega) \leq 1$ for all n and ω. Since bounded monotone sequences have finite limits, $\lim_{n\to\infty} X_n$ exists in the a.s. sense and the limit is less than or equal to one with probability one.

(b) Since a.s. convergence of bounded sequences implies m.s. convergence, $\lim_{n\to\infty} X_n$ also exists in the m.s. sense. To be more precise, a.s. convergence implies convergence in probability (see [1, Proposition 2.7 (a)]), and then we apply [1, Proposition 2.7 (c)] with $Y = 1$.

(c) Since (X_n) converges a.s., it also converges in probability to the same random variable, so $Z = \lim_{n\to\infty} X_n$ a.s. It can be shown that $\mathbb{P}(Z = 1) = 1$. Here is one of several proofs. Let $0 < \varepsilon < 1$. Define $a_0 := 0$ and for $k \geq 1$ that

$$a_k := \frac{a_{k-1} + 1 - \varepsilon}{2}.$$

By induction, we have $a_k = (1 - \varepsilon)(1 - 2^{-k})$. Now define for each $i \geq 1$ the event $A_i := \{U_i \geq 1 - \varepsilon\}$. These events $\{A_i\}_{i\geq 1}$ are independent, and each has probability ε. Therefore, for each $k \geq 1$, the probability that at least k of these events happens is one. Or to be more precise, let B_k be the event that at least k of the events among $\{A_i\}_{i\geq 1}$ happen, and then we have $\mathbb{P}(B_k)=1$. Now note that if at least k of these events $\{A_i\}_{i\geq 1}$ happen, then $Z \geq a_k$. Thus, we have

$$\begin{aligned}\mathbb{P}\left((1-\varepsilon)(1-2^{-k}) \leq Z \leq 1\right) &\geq \mathbb{P}(B_k)\\ &= 1,\end{aligned}$$

which implies that

$$\mathbb{P}\left((1-\varepsilon)(1-2^{-k}) \leq Z \leq 1\right) = 1. \tag{5.1}$$

Since ε can be arbitrarily small and k can be arbitrarily large in (5.1), we can already expect that $\mathbb{P}(Z=1)=1$. The following derivation, which is similar to the arguments in Problem 4, shows that this is indeed true: Pick any sequence $(\varepsilon_\ell)_{\ell\geq 1}$ of numbers in $(0,1)$ that decreases to 0 (say $\varepsilon_\ell = 2^{-\ell}$). Then we have

$$\{Z=1\} = \bigcap_{\ell=1}^{\infty}\bigcap_{k=1}^{\infty}\{(1-\varepsilon_\ell)(1-2^{-k}) \leq Z \leq 1\}. \tag{5.2}$$

Since the intersection of countably many probability-one events also happens with probability one (which can be shown by taking the complement and then using the union bound as what we did in Problem 4), we have from (5.1) and (5.2) that $\mathbb{P}(Z=1)=1$.

Another approach: We first calculate that $\mathbb{E}[X_n|X_{n-1}=v] = v + \frac{(1-v)^2}{2}$. Thus,

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[X_{n-1}] + \frac{\mathbb{E}\left[(1-X_{n-1})^2\right]}{2}\\ &\geq \mathbb{E}[X_{n-1}] + \frac{(1-\mathbb{E}[X_{n-1}])^2}{2},\end{aligned}$$

where the inequality followed from the fact that $\mathbb{E}\left[W^2\right] \geq (\mathbb{E}[W])^2$ for any random variable W. Since $\mathbb{E}[X_n] \to \mathbb{E}[Z]$ (by, e.g., Part (b) and [1, Corollary 2.13]), it follows that

$$\mathbb{E}[Z] \geq \mathbb{E}[Z] + \frac{(1-\mathbb{E}[Z])^2}{2}.$$

So $\mathbb{E}[Z]=1$. In view of the fact $\mathbb{P}(Z\leq 1)=1$, it follows that $\mathbb{P}(Z=1)=1$.

REFERENCES

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf