Random Processes Spring 2026: Homework III

February 19th, 2026

Problem 1. Let $f(x)$ be a function with domain $[0,1]$ such that $f(x) = 1$ if x is a rational number, and $f(x) = 0$ otherwise. What is the Riemann integral of this function? Define the Lebesgue integral as we did in class and compute the Lebesgue integral of the function.

Problem 2. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of $\mathcal{N}(0,1)$ (standard Gaussian) random variables. Define the events

$$A_n = \left\{ |X_n| > \sqrt{2\log n} \right\}.$$

Determine whether the events A_n occur infinitely often. Hint: Use Borel-Cantelli's lemma.

Problem 3. Let $(U_n)_{n\geq 1}$ be independent Unif$[0,1]$ random variables. Define a sequence of random variables $(X_n)_{n\geq 1}$ by

$$X_n(\omega) = n\mathbf{1}_{\{|U_n(\omega)-\omega|\leq 1/n\}},$$

where $\omega \in [0,1]$ and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Determine if the sequence $(X_n)_{n\geq 1}$ converges almost surely. Determine if it converges in probability.

Problem 4. Let $(X_n)_{n\geq 1}$ be a sequence of random variables defined over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let X be another random variable. Suppose that

$$X_n \xrightarrow{P} X, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \quad \mathbb{P}(|X_n - X| > \varepsilon) \to 0.$$

Assume in addition that for every $\varepsilon > 0$, the probabilities of deviations satisfy the *summability condition*

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty.$$

Prove that in this case the sequence also converges in the almost sure sense. Provide an intuition for this result.

Problem 5. Prof. Hajek's text, 2.4, 2.9, 2.10.