ECE 534 SP26 HW1 Solutions

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Problem 1: Countable vs Uncountable

Problem: Show that the rational numbers are countable, while the reals are uncountable.

Solution:

Part 1: Rational numbers are countable.

We need to show that there exists a bijection between $\mathbb{Q}$ and $\mathbb{N}$ (or a subset of $\mathbb{N}$).

Every rational number can be written as $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We can enumerate the rationals by arranging them in a grid:

- Row 0: $\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots$
- Row 1: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$
- Row -1: $\frac{-1}{1}, \frac{-1}{2}, \frac{-1}{3}, \ldots$
- Row 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$
- And so on...

We can traverse this grid diagonally (Cantor's diagonalization method for enumeration):

$$\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{0}{3}, \frac{1}{2}, \frac{-1}{1}, \frac{2}{1}, \frac{-1}{2}, \frac{1}{3}, \ldots$$

Skipping duplicates (e.g., $\frac{0}{2} = \frac{0}{1}$, $\frac{2}{2} = \frac{1}{1}$), we obtain a sequence that includes every rational number exactly once. This establishes a bijection between $\mathbb{Q}$ and $\mathbb{N}$, proving that $\mathbb{Q}$ is countable.

Part 2: Real numbers are uncountable.

We use Cantor's diagonalization argument to prove $\mathbb{R}$ is uncountable. It suffices to show that $(0, 1)$ is uncountable.

Suppose, for contradiction, that $(0, 1)$ is countable. Then we can list all numbers in $(0, 1)$ as a sequence:

$$\begin{aligned} r_1 &= 0.a_{11}a_{12}a_{13}\ldots \\ r_2 &= 0.a_{21}a_{22}a_{23}\ldots \\ r_3 &= 0.a_{31}a_{32}a_{33}\ldots \\ &\vdots \end{aligned}$$

where $a_{ij} \in \{0, 1, \dots, 9\}$ are the decimal digits.

Now construct a new number $x = 0.b_1 b_2 b_3 \dots$ where

$$b_i = \begin{cases} 5 & \text{if } a_{ii} \neq 5, \\ 6 & \text{if } a_{ii} = 5. \end{cases}$$

Then $x \in (0, 1)$ and $x \neq r_i$ for any i because x differs from r_i in the i-th decimal place. This contradicts our assumption that $\{r_1, r_2, r_3, \dots\}$ contains all numbers in $(0, 1)$.

Therefore, $(0, 1)$ is uncountable, and hence $\mathbb{R}$ is uncountable.

Problem 2: Sigma Algebras

Problem: Describe the axioms of σ-algebras and prove that the intersection of two σ-algebras is another σ-algebra.

Solution:

Axioms of σ-algebras:

A collection $\mathcal{F}$ of subsets of Ω is a σ-algebra if it satisfies:

1. **Contains the whole space:** $\Omega \in \mathcal{F}$
2. **Closed under complements:** If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. **Closed under countable unions:** If $A_1, A_2, A_3, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Proof that intersection of two σ-algebras is a σ-algebra:

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras on Ω. Define $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$. We need to verify that $\mathcal{F}$ satisfies the three axioms.

Axiom 1: Since $\Omega \in \mathcal{F}_1$ and $\Omega \in \mathcal{F}_2$, we have $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}$.

Axiom 2: Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras, $A^c \in \mathcal{F}_1$ and $A^c \in \mathcal{F}_2$. Therefore, $A^c \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}$.

Axiom 3: Let $A_1, A_2, A_3, \dots \in \mathcal{F}$. Then for all i, $A_i \in \mathcal{F}_1$ and $A_i \in \mathcal{F}_2$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras:

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \quad \text{and} \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_2.$$

Therefore, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}$.

Since all three axioms are satisfied, $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-algebra.

Problem 3: Random Variables

Problem: Show that if the measurable space is $(\Omega, \mathcal{F})$, with $\mathcal{F} = \{\Omega, \emptyset\}$, then all random variables over this space are constants.

Solution:

Let $X : \Omega \to \mathbb{R}$ be a random variable on $(\Omega, \mathcal{F})$ where $\mathcal{F} = \{\Omega, \emptyset\}$.

Without loss of generality, assume Ω is not empty. Pick an arbitrary $\omega_0 \in \Omega$, and let $c := X(\omega_0)$. By the definition of random variables (Definition 1.9. in Prof. Leveque's notes) and the fact that $\{c\}$ is a Borel set in $\mathbb{R}$, we have

$$X^{-1}(\{c\}) \in \mathcal{F},$$

where $X^{-1}(\{c\}) := \{\omega \in \Omega : X(\omega) = c\}$ denotes the preimage of $\{c\}$ through the map X.

Since $\mathcal{F} = \{\Omega, \emptyset\}$, we now have that $X^{-1}(\{c\})$ is either Ω or $\emptyset$. At the same time, we know that $X^{-1}(\{c\}) \neq \emptyset$ since $\omega_0 \in X^{-1}(\{c\})$. Therefore, we have $X^{-1}(\{c\}) = \Omega$, which implies $X(\omega) = c$ for all $\omega \in \Omega$. These arguments show that X is a constant function.

Problem 4: Continuity of Probability Measures

Problem: Given a measurable space $(\Omega, \mathcal{F})$ and a sequence of nested events $A_1, A_2, A_3, \ldots$ in $\mathcal{F}$ such that $A_{i+1} \subset A_i$ for all i, prove that

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Solution:

Let $A = \bigcap_{i=1}^{\infty} A_i$. We need to show that $\lim_{n\to\infty} P(A_n) = P(A)$.

Define the sequence of sets:

$$B_i = A_i \setminus A_{i+1} = A_i \cap A_{i+1}^c \quad \text{for } i = 1, 2, 3, \ldots$$

The sets B_i are disjoint. To see this, note that if $i < j$, then $A_{i+1} \supseteq A_j$, so

$$B_i \cap B_j = (A_i \setminus A_{i+1}) \cap (A_j \setminus A_{j+1}) \subseteq A_{i+1}^c \cap A_j = \emptyset.$$

Now observe that:

$$A_1 = A \cup \bigcup_{i=1}^{\infty} B_i.$$

This can be verified: an element $\omega \in A_1$ either belongs to all A_i (in which case $\omega \in A$), or there exists a smallest k such that $\omega \notin A_{k+1}$ (in which case $\omega \in B_k$).

Since the B_i are disjoint and $A \cap B_i = \emptyset$ for all i:

$$P(A_1) = P(A) + \sum_{i=1}^{\infty} P(B_i).$$

Also note that:

$$A_n = A \cup \bigcup_{i=n}^{\infty} B_i.$$

Therefore:

$$P(A_n) = P(A) + \sum_{i=n}^{\infty} P(B_i).$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} P(A_n) = P(A) + \lim_{n\to\infty} \sum_{i=n}^{\infty} P(B_i) = P(A) + 0 = P(A).$$

The second equality holds because $\sum_{i=1}^{\infty} P(B_i)$ converges (it equals $P(A_1) - P(A) < \infty$), so the tail sum vanishes.

Therefore, $\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$.

Problem 5: CDFs

Problem: Prove that in general, CDFs do not have to be left continuous but have to be right continuous.

Solution:

Recall that the cumulative distribution function (CDF) of a random variable X is defined as:

$$F_X(x) = P(X \leq x).$$

Part 1: CDFs are right continuous.

We need to show that $\lim_{h\to 0^+} F_X(x+h) = F_X(x)$ for all $x \in \mathbb{R}$.

Consider a decreasing sequence $h_n \to 0^+$ (e.g., $h_n = 1/n$). Define:

$$A_n = \{X \leq x + h_n\}.$$

Then $A_1 \supset A_2 \supset A_3 \supset \cdots$ and

$$\bigcap_{n=1}^{\infty} A_n = \{X \leq x\}.$$

By the continuity of probability measures (Problem 4):

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right),$$

$$\lim_{n\to\infty} F_X(x + h_n) = F_X(x).$$

Since this holds for any sequence $h_n \to 0^+$, we have $\lim_{h\to 0^+} F_X(x+h) = F_X(x)$.

Part 2: CDFs are not necessarily left continuous.

To show this, we provide a counterexample. Consider a discrete random variable:

$$X = \begin{cases} 0 & \text{with probability } 1/2, \\ 1 & \text{with probability } 1/2. \end{cases}$$

The CDF is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1/2 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

At $x = 1$:

$$\lim_{h\to 0^-} F_X(1+h) = \lim_{h\to 0^-} F_X(1-|h|) = F_X(1^-) = 1/2,$$

$$F_X(1) = 1.$$

Since $F_X(1^-) = 1/2 \neq 1 = F_X(1)$, the CDF is not left continuous at $x = 1$.

In general, F_X has a jump discontinuity at any point x_0 where $P(X = x_0) > 0$. The size of the jump is exactly $P(X = x_0)$.

Problem 6: More on RVs

Problem: Let X be a nonnegative extended real-valued random variable, $X : \Omega \to [0, \infty]$, such that

$$\lim_{n\to\infty} P(X \geq n) = 0.$$

Show that $P(X < \infty) = 1$.

Solution:

Define the events:

$$A_n = \{X \geq n\} = \{\omega \in \Omega : X(\omega) \geq n\}.$$

We are given that $\lim_{n\to\infty} P(A_n) = 0$.

Note that the sequence $\{A_n\}$ is decreasing: $A_1 \supset A_2 \supset A_3 \supset \cdots$ because if $X(\omega) \geq n+1$, then $X(\omega) \geq n$.

Define:

$$A = \bigcap_{n=1}^{\infty} A_n = \{X = \infty\}.$$

An element $\omega \in A$ if and only if $\omega \in A_n$ for all n, which means $X(\omega) \geq n$ for all $n \in \mathbb{N}$. This is equivalent to $X(\omega) = \infty$.

By the continuity of probability measures (Problem 4):

$$P(A) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = 0.$$

Therefore:

$$P(X = \infty) = 0.$$

Since X takes values in $[0, \infty]$, we have:

$$P(X < \infty) = 1 - P(X = \infty) = 1 - 0 = 1.$$

Problem 7: Discrete Random Variables

Problem: Let X be a binomial random variable with parameters (n, p), where $p \in (0, 1)$. Find $P\{X \text{ even}\}$, $E[X]$, and $E[X^2 + 3]$.

Solution:

Recall that for $X \sim \text{Binomial}(n, p)$:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Part 1: Find $P\{X \text{ even}\}$.

We use the binomial theorem. Consider:

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k},$$

$$(p-q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k (-q)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (-1)^{n-k},$$

where $q = 1 - p$.

Adding these two equations:

$$(p+q)^n + (p-q)^n = 2 \sum_{k \text{ even}} \binom{n}{k} p^k q^{n-k}.$$

Since $p + q = 1$:

$$1 + (2p-1)^n = 2 \sum_{k \text{ even}} \binom{n}{k} p^k q^{n-k} = 2P(X \text{ even}).$$

Therefore:

$$P(X \text{ even}) = \frac{1 + (2p-1)^n}{2}.$$

Part 2: Find $E[X]$.

For a binomial random variable with parameters (n, p):

$$E[X] = np.$$

This is a standard result. To derive it:

$$\begin{aligned}
E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\
&= np(p + (1-p))^{n-1} = np.
\end{aligned}$$

Part 3: Find $E[X^2 + 3]$.
By linearity of expectation:

$$E[X^2 + 3] = E[X^2] + 3.$$

For a binomial random variable, we can derive $E[X^2]$ using the fact that:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

For $X \sim \mathrm{Binomial}(n, p)$, we have $\mathrm{Var}(X) = np(1-p)$. Therefore:

$$E[X^2] = \mathrm{Var}(X) + (E[X])^2 = np(1-p) + (np)^2 = np(1-p) + n^2p^2.$$

Thus:

$$E[X^2 + 3] = np(1-p) + n^2p^2 + 3 = np - np^2 + n^2p^2 + 3.$$

We can also write this as:

$$E[X^2 + 3] = n^2p^2 + np(1-p) + 3 = n^2p^2 + npq + 3,$$

where $q = 1 - p$.

Problem 8: Continuous Random Variables

Problem: Let X be a Gaussian random variable with parameters (μ, σ^2). Find $P\{X^2 \leq 3\}$. Can you come up with simple bounds on this probability?

Solution:
For $X \sim \mathcal{N}(\mu, \sigma^2)$, the PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Part 1: Find $P\{X^2 \leq 3\}$.
The event $\{X^2 \leq 3\}$ is equivalent to $\{-\sqrt{3} \leq X \leq \sqrt{3}\}$. Therefore:

$$P(X^2 \leq 3) = P(-\sqrt{3} \leq X \leq \sqrt{3}) = P\left(\frac{-\sqrt{3}-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{\sqrt{3}-\mu}{\sigma}\right).$$

Let $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. Then:

$$P(X^2 \leq 3) = \Phi\left(\frac{\sqrt{3}-\mu}{\sigma}\right) - \Phi\left(\frac{-\sqrt{3}-\mu}{\sigma}\right),$$

where $\Phi(\cdot)$ is the standard normal CDF:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

Alternatively, we can write:

$$P(X^2 \leq 3) = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$$

Part 2: Simple bounds on this probability.

Method 1: Markov's inequality

Since X^2 is a non-negative random variable, we can apply Markov's inequality to get

$$\begin{aligned} P(X^2 > 3) &\leq \frac{1}{3} E[X^2] \\ &= \frac{1}{3}(\sigma^2 + \mu^2). \end{aligned}$$

So we get an lower bound on $P(X^2 \leq 3)$ by

$$P(X^2 \leq 3) \geq 1 - \frac{1}{3}(\sigma^2 + \mu^2),$$

which is a meaningful bound only when $\frac{1}{3}(\sigma^2 + \mu^2) < 1$.

Method 2: Chebyshev's inequality

Chebyshev's inequality states that for any random variable with finite mean and variance:

$$P(|X - \mu| \geq k) \leq \frac{\text{Var}(X)}{k^2}.$$

This bound depends on the relationship between $|\mu|$ and $\sqrt{3}$. Intuitively, when μ is between $-\sqrt{3}$ and $\sqrt{3}$, then the interval $[-\sqrt{3}, \sqrt{3}]$ contains the mean of X, and thus we can expect a lower bound of the probability that X lies within this interval (which is precisely $P(X^2 \leq 3)$). Otherwise, we can expect an upper bound.

To be more precise, we divide our discussion based on the value of μ:

1. If $\mu > \sqrt{3}$, then

$$\begin{aligned} P(X^2 \leq 3) &= P(-\sqrt{3} \leq X \leq \sqrt{3}) \\ &\leq P(X \leq \sqrt{3}) \\ &= \frac{1}{2} P(|X - \mu| \geq \mu - \sqrt{3}) \\ &\leq \frac{\sigma^2}{2(\mu - \sqrt{3})^2}. \end{aligned}$$

2. If $\mu = \sqrt{3}$, then we have

$$\begin{aligned} P(X^2 \leq 3) &= \frac{1}{2} P(|X - \mu| \leq 2\sqrt{3}) \\ &\geq \frac{1}{2} - \frac{\sigma^2}{24}. \end{aligned}$$

3. If $0 < \mu < \sqrt{3}$, then we have

$$\begin{aligned} P(X^2 \leq 3) &\geq P(|X - \mu| \leq \sqrt{3} - \mu) \\ &\geq 1 - \frac{\sigma^2}{(\sqrt{3} - \mu)^2}. \end{aligned}$$

4. If $\mu = 0$, then

$$\begin{aligned} P(X^2 \leq 3) &= P(|X - \mu|^2 \leq 3) \\ &\geq 1 - \frac{\sigma^2}{3}. \end{aligned}$$

5. If $-\sqrt{3} < \mu < 0$, then similar to the calculation for the case $0 < \mu < \sqrt{3}$, we can get

$$P(X^2 \leq 3) \geq 1 - \frac{\sigma^2}{(\sqrt{3} - |\mu|)^2}.$$

6. If $\mu = -\sqrt{3}$, then similar to the calculation for the case $\mu = \sqrt{3}$, we can get

$$P(X^2 \leq 3) \geq \frac{1}{2} - \frac{\sigma^2}{24}.$$

7. If $\mu < -\sqrt{3}$, then similar to the calculation for the case $\mu > \sqrt{3}$, we can get

$$P(X^2 \leq 3) \leq \frac{\sigma^2}{2(|\mu| - \sqrt{3})^2}.$$

In summary, we have

$$P(X^2 \leq 3) \begin{cases} \leq \frac{\sigma^2}{2(|\mu| - \sqrt{3})^2}, \text{ if } |\mu| < \sqrt{3}, \\ \geq \frac{1}{2} - \frac{\sigma^2}{24}, \text{ if } |\mu| = \sqrt{3}, \\ \geq 1 - \frac{\sigma^2}{(\sqrt{3} - |\mu|)^2}, \text{ if } 0 \leq |\mu| < \sqrt{3}. \end{cases}$$

Note: This bound is quite loose for normal distributions.

Method 3: Using Gaussian tail bounds

For $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$P(|X - \mu| > k\sigma) \leq 2e^{-k^2/2} \quad \text{(Gaussian tail bound)},$$

the proof of which can be obtained by applying Markov's inequality to $e^{\lambda Z}$ for $Z \sim \mathcal{N}(0, 1)$ and some suitable choice of λ (which is essentially the Chernoff bound).

We can repeat the same procedure as we did for the Chebyshev's inequality to obtain tighter bounds on $P(X^2 \leq 3)$ depending on the value of $|\mu|$.

Simple numerical bounds (specific case $\mu = 0, \sigma = 1$):

If $\mu = 0$ and $\sigma = 1$ (i.e. if X itself is standard normal):

$$P(X^2 \leq 3) = P(|X| \leq \sqrt{3}) = 2\Phi(\sqrt{3}) - 1 \approx 2(0.9133) - 1 = 0.8266.$$

Using the 68-95-99.7 rule: $P(|X| \leq 1.73) \approx 0.83$ is between one and two standard deviations, which matches our calculation.