ECE 534 Recitation

Problem 1

Consider the Markov chain with three states, $S = \{1, 2, 3\}$, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

a. Draw the state transition diagram for this chain.
b. If we know $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

a. The state transition diagram is shown in Figure 11.6

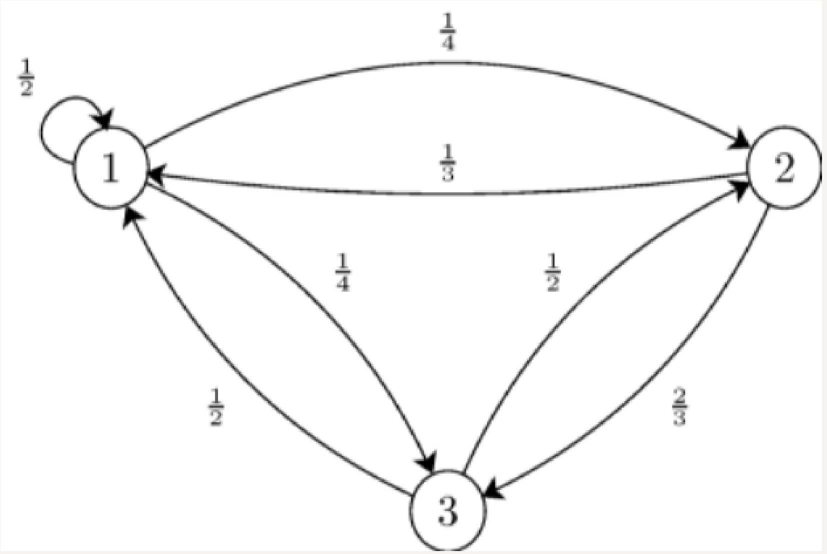

Figure 11.6 - A state transition diagram.

b. First, we obtain

$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{12}. \end{aligned}$$

Problem 5

Consider the Markov chain shown in Figure 11.20.

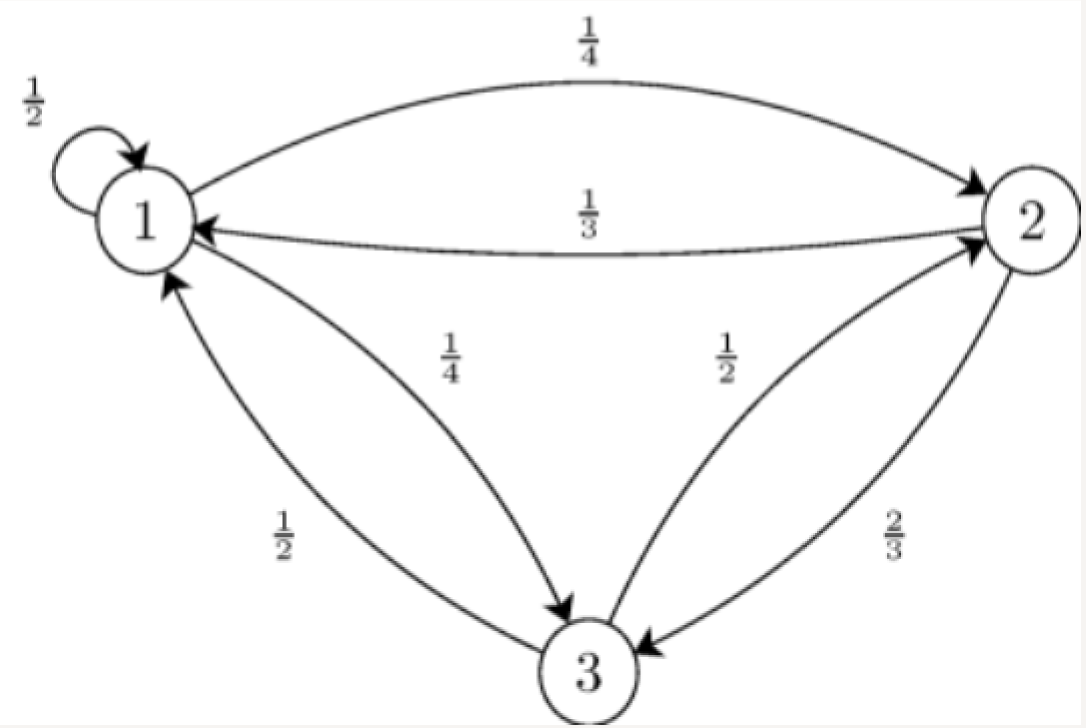

Figure 11.20 - A state transition diagram.

a. Is this chain irreducible?
b. Is this chain aperiodic?
c. Find the stationary distribution for this chain.
d. Is the stationary distribution a limiting distribution for the chain?

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.
b. The chain is aperiodic since there is a self-transition, i.e., $p_{11} > 0$.
c. To find the stationary distribution, we need to solve

$$\begin{aligned} \pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\ \pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3, \\ \pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2, \\ \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned}$$

   We find

$$\pi_1 \approx 0.457, \ \pi_2 \approx 0.257, \ \pi_3 \approx 0.286$$

d. The above stationary distribution is a limiting distribution for the chain because the chain is irreducible and aperiodic.

Problem 6

Consider the Markov chain shown in Figure 11.21. Assume that $\frac{1}{2} < p < 1$. Does this chain have a limiting distribution? For all $i, j \in \{0, 1, 2, \cdots\}$, find

$$\lim_{n \to \infty} P(X_n = j | X_0 = i).$$

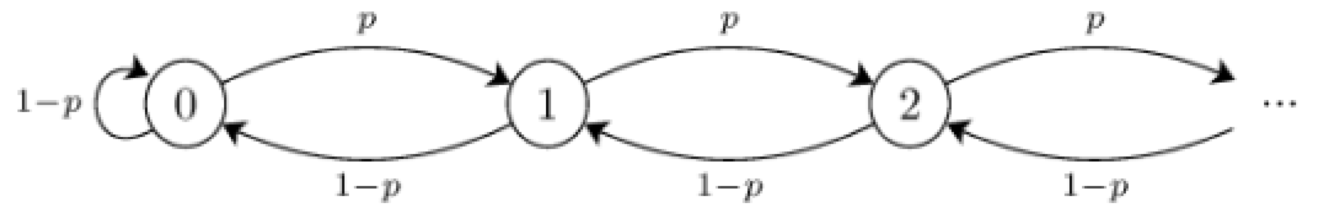

Figure 11.21 - A state transition diagram.

This chain is irreducible since all states communicate with each other. It is also aperiodic since it includes a self-transition, $P_{00} > 0$. Let's write the equations for a stationary distribution. For state 0, we can write

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1,$$

which results in

$$\pi_1 = \frac{p}{1-p}\pi_0.$$

For state 1, we can write

$$\begin{aligned}\pi_1 &= p\pi_0 + (1-p)\pi_2 \\ &= (1-p)\pi_1 + (1-p)\pi_2,\end{aligned}$$

which results in

$$\pi_2 = \frac{p}{1-p}\pi_1.$$

Similarly, for any $j \in \{1, 2, \cdots\}$, we obtain

$$\pi_j = \alpha\pi_{j-1},$$

where $\alpha = \frac{p}{1-p}$. Note that since $\frac{1}{2} < p < 1$, we conclude that $\alpha > 1$. We obtain

$$\pi_j = \alpha^j\pi_0, \quad \text{for } j = 1, 2, \cdots.$$

Finally, we must have

$$\begin{aligned}1 &= \sum_{j=0}^{\infty}\pi_j \\ &= \sum_{j=0}^{\infty}\alpha^j\pi_0, \qquad (\text{where } \alpha > 1) \\ &= \infty\pi_0.\end{aligned}$$

Therefore, the above equation cannot be satisfied if $\pi_0 > 0$. If $\pi_0 = 0$, then all π_j's must be zero, so they cannot sum to 1. We conclude that there is no stationary distribution. This means that either all states are transient, or all states are null recurrent. In either case, we have

$$\lim_{n\to\infty} P(X_n = j|X_0 = i) = 0, \text{ for all } i, j.$$