ECE 534 Recitation

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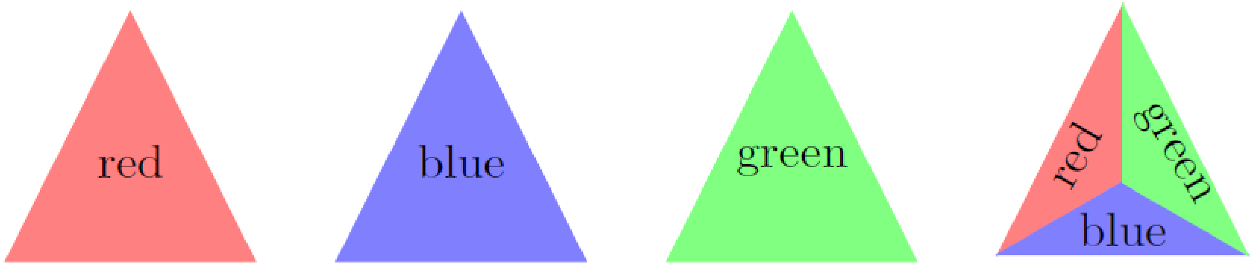
2/3/2022

A Few Announcements

- Homework 1 is out. Check course website
- All claims mentioned in course does not require proofs
- HW1 Problem 5 are from the textbook
- Quiz should only cover basic probability theory and sigma-algebra

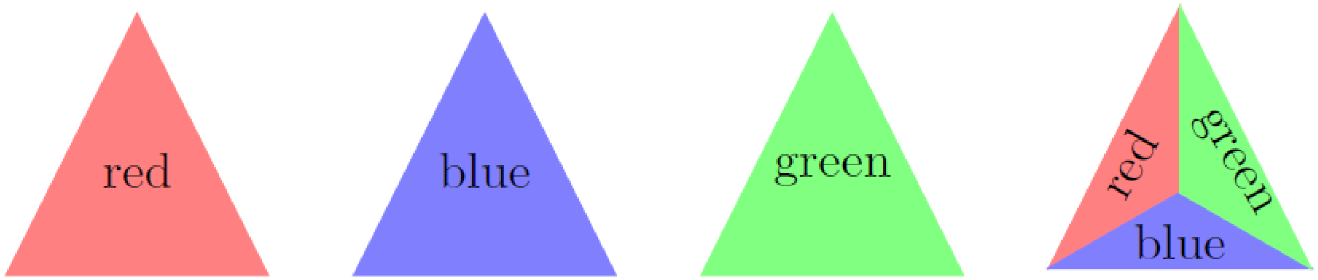
A tetrahedron has four faces, which are painted as follows: one side all red, one side all blue, one side all green, and one side with red, blue and green.



Toss the tetrahedron randomly and the face that lands on the floor is equally likely among the four. Define the following events:

$$R = \{\text{the face that hits the floor has red color}\}$$
$$B = \{\text{the face that hits the floor has blue color}\}$$
$$G = \{\text{the face that hits the floor has green color}\}$$

(a) Compute the probabilities: $P[R], P[G], P[B]$.
Solution: $P[R] = P[G] = P[B] = 1/2$.



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$$P[R] = P[G] = P[B] = 1/2.$$

(b) Are the events R, G, B pairwise independent? Justify your answer by calculations.
Solution: $P[RG] = P[GB] = P[BR] = P[\text{4th face}] = 1/4$. So they are pairwise independent.

(c) Are the events R, G, B independent? Justify your answer by calculations.
Solution: $P[RGB] = P[\text{4th face}] = 1/4 \neq P[R]P[G]P[B]$. So they are not independent.

[8+6+10+6 points] The joint pmf of two discrete-type random variables is as shown in the table below

	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	0.2	0	0.1	0.1
$Y=2$	0	0.25	0.15	0.05
$Y=3$	0.05	0	0	0.1

(a) Find the marginal pmfs of X and Y.

Solution: The marginal pmf of X is given by the column sums. Thus

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = 0.25.$$

The marginal pmf of Y is given by the row sums. Thus

$$p_Y(1) = 0.4, \quad p_Y(2) = 0.45, \quad p_Y(3) = 0.15.$$

	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	0.2	0	0.1	0.1
$Y=2$	0	0.25	0.15	0.05
$Y=3$	0.05	0	0	0.1

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = 0.25.$$

$$p_Y(1) = 0.4, \quad p_Y(2) = 0.45, \quad p_Y(3) = 0.15.$$

(b) Are X and Y independent? Justify your answer.
Solution: No. For example, $p_{X,Y}(1,1) = 0.2$ and $p_X(1)p_Y(1) = 0.25 \times 0.4 = 0.1$. Thus $p_{X,Y}(1,1) \neq p_X(1)p_Y(1)$.

	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	0.2	0	0.1	0.1
$Y=2$	0	0.25	0.15	0.05
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$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = 0.25.$$

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(c) Find the conditional pmf $p_{Y|X}(v|4)$.

Solution:

$$p_{Y|X}(v|4) = \frac{p_{X,Y}(4,v)}{p_X(4)} = \begin{cases} \frac{0.1}{0.25} = 0.4 & \text{if } v = 1 \\ \frac{0.05}{0.25} = 0.2 & \text{if } v = 2 \\ \frac{0.1}{0.25} = 0.4 & \text{if } v = 3 \end{cases}$$

	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	0.2	0	0.1	0.1
$Y=2$	0	0.25	0.15	0.05
$Y=3$	0.05	0	0	0.1

(d) Find $P\{X < Y\}$.

Solution:

$$P\{X < Y\} = p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(2,3) = 0.05.$$

[10 points] Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1$ and $\sigma^2 = 4$. Find $P(|X| < 1)$. Express your solution in terms of the Q function.

Solution:

$$\begin{aligned} P(|X| < 1) &= P(-1 < X < 1) = P(X < 1) - P(X < -1) = \\ &= P\left(\frac{X-\mu}{\sigma} < \frac{1-1}{2}\right) - P\left(\frac{X-\mu}{\sigma} < \frac{-1-1}{2}\right) \\ &= P(Z < 0) - P(Z < -1), \end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$. Using the symmetry properties of the Q function and the given hint:

$$P(|X| < 1) = Q(0) - Q(1) = \frac{1}{2} - Q(1).$$

If Y is a Gaussian random variable with mean μ and variance σ^2, then $X = \dfrac{Y-\mu}{\sigma}$ is standard normal and

$$P(Y > y) = P(X > x) = Q(x)$$

where $x = \dfrac{y-\mu}{\sigma}$.

Example 1.8 Suppose a vehicle is traveling in a straight line at speed a, and that a random direction is selected, subtending an angle Θ from the direction of travel which is uniformly distributed over the interval $[0, \pi]$. See Figure 1.7. Then the effective speed of the vehicle in the random direction is $B = a\cos(\Theta)$.

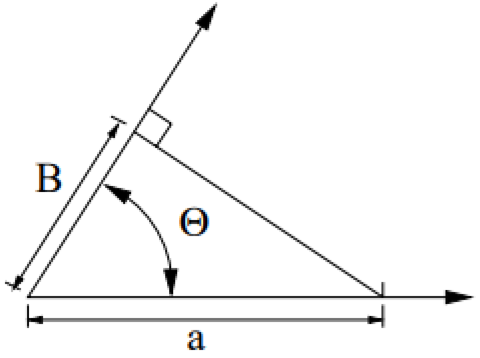

Figure 1.7 Direction of travel and a random direction.

Let us find the pdf of B.

The range of $a\cos(\theta)$, as θ ranges over $[0, \pi]$, is the interval $[-a, a]$. Therefore, $F_B(c) = 0$ for $c \leq -a$ and $F_B(c) = 1$ for $c \geq a$. Let now $-a < c < a$. Then, because cos is monotone nonincreasing on the interval $[0, \pi]$,

$$\begin{aligned} F_B(c) &= P\{a\cos(\Theta) \leq c\} = P\left\{\cos(\Theta) \leq \frac{c}{a}\right\} \\ &= P\left\{\Theta \geq \cos^{-1}\left(\frac{c}{a}\right)\right\} \\ &= 1 - \frac{\cos^{-1}\left(\frac{c}{a}\right)}{\pi}. \end{aligned}$$

Therefore, because $\cos^{-1}(y)$ has derivative, $-(1-y^2)^{-\frac{1}{2}}$,

$$f_B(c) = \begin{cases} \frac{1}{\pi\sqrt{a^2-c^2}} & \mid c \mid < a \\ 0 & \mid c \mid > a \end{cases}.$$

[6+8+12 points] Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 1$ bus per minute. Bob arrives at the bus stop at 12 noon and wants to catch the next bus. For all parts of this question, you can leave your answer in terms of e, the base of natural logarithm, e.g. $2e^{-1}$.

(a) What is the probability that exactly three buses arrive within one minute?

Solution: Denote the number of buses arriving within one minute by N_1, which has a Poisson distribution with mean $\lambda \times 1 = 1$, hence

$$P(N_1 = 3) = \frac{e^{-1}1^3}{3!} = \frac{e^{-1}}{6}.$$

Notation	$\mathrm{Pois}(\lambda)$
Parameters	$\lambda \in (0, \infty)$ (rate)
Support	$k \in \mathbb{N}_0$ (Natural numbers starting from 0)
PMF	$\frac{\lambda^k e^{-\lambda}}{k!}$

Mean	λ
Median	$\approx \lfloor \lambda + 1/3 - 0.02/\lambda \rfloor$
Mode	$\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$
Variance	λ

[6+8+12 points] Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 1$ bus per minute. Bob arrives at the bus stop at 12 noon and wants to catch the next bus. For all parts of this question, you can leave your answer in terms of e, the base of natural logarithm, e.g. $2e^{-1}$.

(b) Given Bob is still waiting at the bus stop at 12:05pm (i.e., no bus arrived), what is the probability that he will still be waiting at 12:10pm?

Solution: Denote the time between 12 noon and the arrival of the first bus by T_1, which has an exponential distribution with rate $\lambda = 1$, hence using memoryless property of exponential distribution,

$$P(T_1 > 10 | T_1 > 5) = P(T_1 > 5) = e^{-5}.$$

[6+8+12 points] Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 1$ bus per minute. Bob arrives at the bus stop at 12 noon and wants to catch the next bus. For all parts of this question, you can leave your answer in terms of e, the base of natural logarithm, e.g. $2e^{-1}$.

(c) Given a total of two buses arrived between 12pm and 12:02pm, what's the probability that exactly one bus arrived between 12pm and 12:01pm?

Solution: Denote the number of buses arriving between 12pm and 12:02pm by N_2, the number arriving between 12pm and 12:01pm by N_1 and the number arriving between 12:01pm and 12:02pm by M_1. The random variable N_2 has a Poisson distribution with mean $2\lambda = 2$. The random variables N_1 and M_1 are independent and each has a Poisson distribution with mean $\lambda = 1$. Hence,

$$P(N_1 = 1|N_2 = 2) = \frac{P(N_1 = 1, M_1 = 1)}{P(N_2 = 2)} = \frac{P(N_1 = 1)P(M_1 = 1)}{P(N_2 = 2)} = \frac{e^{-1}e^{-1}}{\frac{e^{-2}2^2}{2!}} = 1/2.$$

[8+12 points] Suppose a random variable X has the following pdf:

$$f_X(u) = \begin{cases} c(1-u) & 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) Find the constant c

Solution: The total probability should be equal to 1,

$$\int_{-\infty}^{\infty} f_X(u)du = c\int_0^1 (1-u)du = \frac{1}{2}c = 1$$

Therefore, $c = 2$.

[8+12 points] Suppose a random variable X has the following pdf:

$$f_X(u) = \begin{cases} c(1-u) & 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

(b) Let Y denote the standardized version of X, i.e., Y is a linearly scaled version of X that has mean 0 and variance 1. What is the PDF of Y?

Solution: The mean of the random variable X

$$\mu = E[X] = 2\int_0^1 u(1-u)du = \frac{1}{3}$$

2

and

$$E[X^2] = 2\int_0^1 u^2(1-u)du = \frac{1}{6}$$

Therefore the standard deviation for X is $\sigma = \sqrt{E[X^2] - E[X]^2} = \frac{1}{3\sqrt{2}}$. The support for Y is therefore $[(0-1/3)3\sqrt{2}, (1-1/3)3\sqrt{2}] = [-\sqrt{2}, 2\sqrt{2}]$. Use the linear scaling property of the pdf that if $Y = aX + b$, then $f_Y(v) = \frac{1}{a} f_X((v-b)/a)$. Since $Y = \frac{X-\mu}{\sigma}$, $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$,

$$f_Y(v) = \sigma f_X(\sigma v + \mu) = \begin{cases} 2(1-(v/3\sqrt{2}+1/3))/3\sqrt{2} = \frac{2\sqrt{2}}{9} - \frac{1}{9}v & -\sqrt{2} \leq v \leq 2\sqrt{2} \\ 0 & \text{else} \end{cases}$$

[14 points] Let $X_1 \sim \text{Exp}(\lambda)$ and $X_2 \sim \text{Exp}(3\lambda)$ be independent random variables. Let also $Y = \min\{X_1, X_2\}$. Suppose that $Y = 3$ is observed. Find the Maximum Likelihood estimate of λ.

Solution:

$$\begin{aligned} 1 - F_Y(y) = P(Y > y) = P(\min\{X_1, X_2\} > y) &= P(X_1 > y)P(X_2 > y) \\ &= e^{-\lambda y} e^{-3\lambda y} = e^{-4\lambda y}, \quad y \geq 0. \end{aligned}$$

Therefore, $Y \sim \text{Exp}(4\lambda)$. Moreover,

$$f_Y(y) = 4\lambda e^{-4\lambda y}, \quad y \geq 0.$$

For $Y = 3$, we have

$$L(\lambda) = f_Y(3) = 4\lambda e^{-12\lambda}.$$

Setting the derivative with respect to λ of the likelihood function to zero and solving for λ we obtain:

$$\lambda_{\text{ML}} = \frac{1}{12}.$$

Since the derivative reaches a maximum at $\lambda_{\text{ML}} = \frac{1}{12}$, this is the ML estimate.

Which of the following statements are correct?

a) $A \subset B, B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$

b) $B \subset C, B \in \mathcal{F} \Rightarrow C \in \mathcal{F}$

c) $A \in \mathcal{F}, \mathcal{F} \subset \mathcal{G} \Rightarrow A \in \mathcal{G}$

d) $A \in \mathcal{F}, \mathcal{F} \supset \mathcal{G} \Rightarrow A \in \mathcal{G}$

e) $\mathcal{F} \subset \mathcal{G}, \mathcal{F} \subset \mathcal{H} \Rightarrow \mathcal{G} \subset \mathcal{H}$

f) $\mathcal{F} \subset \mathcal{G}, \mathcal{G} \subset \mathcal{H} \Rightarrow \mathcal{F} \subset \mathcal{H}$

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Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Which of the following (in-)equalities hold for <u>every</u> pair of events $A, B \in \mathcal{F}$?

a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$ if $A \cap B = \emptyset$

b) $\mathbb{P}(A \setminus B) \leq \mathbb{P}(B)$

c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

d) $\mathbb{P}(A) \cdot \mathbb{P}(A^c) = 1$

e) $\mathbb{P}(A) \cdot \mathbb{P}(A^c) \leq \frac{1}{4}$

f) $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

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- Some other quiz questions about sigma-algebra and probability axioms:
- https://moodle.epfl.ch/pluginfile.php/2873138/mod_resource/content/1/APA_quiz1.pdf
- https://moodle.epfl.ch/pluginfile.php/2873325/mod_resource/content/1/APA_quiz2.pdf