ECE 534 recitation 0303

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1 Other approaches for part of HW2 problems

- **2.9 (c)** First let us consider the conditional expectation $\mathbb{E}[X_n|X_{n-1}=v]$. By the recursion formula, we have the following conditional distribution for X_n:

$$X_n = \begin{cases} v & \text{if } U_n \le v, \\ \frac{v+U_n}{2} & \text{if } U_n > v. \end{cases}$$

Then we have have the conditional expectation as

$$\mathbb{E}[X_n|X_{n-1}=v] = \int_0^v v dt + \int_v^1 \frac{v+t}{2} dt = \frac{(v+1)^2}{4}.$$

Then $\mathbb{E}[X_n] = \mathbb{E}\left[\mathbb{E}[X_n|X_{n-1}]\right] = \mathbb{E}\left[\frac{(X_{n-1}+1)^2}{4}\right]$. Since $f(v) = \frac{(v+1)^2}{4}$ is a convex function, we have $\mathbb{E}[X_n] = \mathbb{E}\left[\frac{(X_{n-1}+1)^2}{4}\right] \ge \frac{(\mathbb{E}[X_{n-1}]+1)^2}{4}$. From the m.s. convergence in (b) we know $\mathbb{E}[X_n] \to \mathbb{E}[Z]$, so we have $\mathbb{E}[Z] \ge \frac{(\mathbb{E}[Z]+1)^2}{4}$. This leads to $\mathbb{E}[Z]=1$. From (a) we know that $\mathbb{P}(Z \le 1) = 1$. This implies that $\mathbb{P}(Z=1)=1$.

- **2.23 (b)** From (a) we have the CDF $F(c) = (1-n^{-c})^n$ when $c > 0$. Consider $\ln F(c) = n \ln(1-n^{-c}) = \frac{\ln(1-n^{-c})}{1/n}$. When $n \to +\infty$, both numerator and denominator will go to 0. After checking the continuity of both functions, we can use the L'Hôpital's rule:

$$\lim_{n\to+\infty} \frac{\ln(1-n^{-c})}{1/n} = \lim_{n\to+\infty} \frac{-cn}{n^c - 1}.$$

When $c > 1$, RHS will converge to 0; when $c = 1$, RHS will converge to -1; when $0 < c < 1$, RHS will go to $-\infty$. Note that these are the results for $\ln F(c)$. For $F(c)$ we have

$$F(c) = \begin{cases} 0 & \text{if } c < 1 \\ e^{-1} & \text{if } c = 1 \\ 1 & \text{if } c > 1 \end{cases}$$

- **2.19** To calculate the Chernoff bound, we first compute the moment generating function as $M(\theta) = \mathbb{E}(e^{\theta X_1}) = 0.5e^{-\theta} + 0.1 + 0.4e^{\theta}$, and the function $l(a)$ is defined as $l(a) = \sup_\theta \theta a - \ln M(\theta)$. Since in our case $a = 0$, so we are trying to calculate the value of $l(0)$. By definition, $l(0) = \sup_\theta - \ln M(\theta)$. Since $\ln(\cdot)$ is an increasing function, it is equivalent to find $\inf_\theta M(\theta)$. By taking $M'(\theta) = 0$, we have $\theta = \ln\left(\frac{\sqrt{5}}{2}\right)$. Therefore, $l(0) = -\ln M\left(\ln\left(\frac{\sqrt{5}}{2}\right)\right)$. By Chernoff bound,

$$\begin{aligned}\mathbb{P}(S \geq 0) \leq \exp(-nl(0)) &= \exp\left(n \ln M\left(\ln\left(\frac{\sqrt{5}}{2}\right)\right)\right) \\ &= \left(\frac{1}{10} + \frac{2\sqrt{5}}{5}\right)^{100}\end{aligned}$$