ECE 534 FA26 HW6 Solutions

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The solutions to even-numbered problems can be found in [1, Chapter 12].

I. Solutions to Odd-Numbered Problems

4.35 Mean hitting time for a discrete-time Markov process

(a) The one-step transition probability matrix is

$$P = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.8 & 0 & 0.2 \\ 0 & 0.4 & 0.6 \end{pmatrix}.$$

(b) To find the equilibrium probability distribution $\pi = (\pi_1, \pi_2, \pi_3)$, we solve $\pi P = \pi$ subject to $\pi_1 + \pi_2 + \pi_3 = 1$. From $\pi P = \pi$, we obtain

$$0.4\pi_1 = 0.8\pi_2, \qquad 0.2\pi_2 = 0.4\pi_3.$$

Thus,

$$\pi_2 = \frac{\pi_1}{2}, \qquad \pi_3 = \frac{\pi_2}{2} = \frac{\pi_1}{4}.$$

Since the probabilities sum to one,

$$\pi_1 + \frac{\pi_1}{2} + \frac{\pi_1}{4} = 1,$$

which gives

$$\boxed{\pi = \left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)}.$$

(c) Recall that

$$a_i = E[\tau \mid X_0 = i], \qquad \tau = \min\{k \geq 0 : X_k = 3\}.$$

Clearly, $a_3 = 0$.
By considering the possible values of X_1 and using the Markov property, we obtain

$$a_1 = 1 + 0.6a_1 + 0.4a_2, \tag{1}$$

$$a_2 = 1 + 0.8a_1 + 0.2a_3. \tag{2}$$

To see why (1) holds, starting from State 1, after one step (which corresponds to the additive term 1 in (1)), we have:

- With probability 0.6, the Markov chain stays at State 1, in which case it takes a_1 more steps (in expectation) to reach State 3.
- With probability 0.4, the Markov chain moves to State 2, in which case it takes a_2 more steps (in expectation) to reach State 3.

A similar explanation holds for (2).
Using $a_3 = 0$, these equations simplify to

$$a_1 = a_2 + 2.5, \qquad a_2 = 1 + 0.8a_1.$$

Substituting the second equation into the first, we obtain

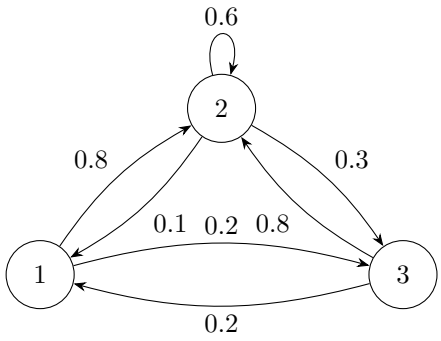
$$a_1 = 3.5 + 0.8a_1,$$

which yields

$$\boxed{a_1 = 17.5, \qquad a_2 = 15}.$$

4.41 A state space reduction preserving the Markov property

(a) The transition probability diagram is shown below.



To find the equilibrium probability distribution $\pi = (\pi_1, \pi_2, \pi_3)$, we solve $\pi P = \pi$, which can be written as

$$-\pi_1 + 0.1\pi_2 + 0.2\pi_3 = 0, \tag{3}$$
$$0.8\pi_1 - 0.4\pi_2 + 0.8\pi_3 = 0, \tag{4}$$
$$0.2\pi_1 + 0.3\pi_2 - \pi_3 = 0. \tag{5}$$

Adding four times (3) to (4) yields

$$-3.2\pi_1 + 1.6\pi_3 = 0,$$

or $\pi_3 = 2\pi_1$.
Substituting this into (3) yields

$$-0.6\pi_1 + 0.1\pi_2 = 0,$$

or $\pi_2 = 6\pi_1$.
Thus,

$$\pi = (\pi_1, 6\pi_1, 2\pi_1).$$

Since the probabilities must sum to one, we obtain

$$\boxed{\pi = \left(\frac{1}{9}, \frac{2}{3}, \frac{2}{9}\right)}.$$

(b) Since $p_{1,2} = p_{3,2} = 0.8$, the probability of transitioning to state 2 is the same whether the current state is 1 or 3. Consequently, states 1 and 3 can be grouped together while preserving the Markov property.
Specifically, take

$$f(1) = f(3) = a, \qquad f(2) = b,$$

where $a \neq b$, and define $Y_k = f(X_k)$.
For both $i = 1$ and $i = 3$, we have

$$P(Y_{k+1} = b \mid X_k = i) = P(X_{k+1} = 2 \mid X_k = i) = 0.8.$$

Therefore, the transition probabilities from the group $\{1, 3\}$ do not depend on which state in that group the original process currently occupies. Hence, by the Markov property of X, the conditional distribution of Y_{k+1} given the history of Y depends only on Y_k.
Thus, Y is a time-homogeneous Markov process with state space $\{a, b\}$.
The one-step transition probabilities are

$$P(Y_{k+1} = a \mid Y_k = a) = 0.2,$$
$$P(Y_{k+1} = b \mid Y_k = a) = 0.8,$$
$$P(Y_{k+1} = a \mid Y_k = b) = 0.1 + 0.3 = 0.4,$$
$$P(Y_{k+1} = b \mid Y_k = b) = 0.6.$$

Therefore, with states ordered as (a, b), the transition probability matrix of Y is

$$\boxed{P_Y = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}}.$$

II. Solutions to the Advanced Problem Set on Markov Chains

Solution.

1) *Detailed balance.* For $0 \le k < N$,

$$\pi(k)P(k,k+1) = 2^{-N}\binom{N}{k}\frac{N-k}{N} = 2^{-N}\binom{N}{k+1}\frac{k+1}{N} = \pi(k+1)P(k+1,k),$$

using $\binom{N}{k}(N-k) = \binom{N}{k+1}(k+1)$. Detailed balance implies stationarity: $\sum_k \pi(k)P(k,j) = \sum_k \pi(j)P(j,k) = \pi(j)$. The chain is irreducible on a finite set (one can move $k \to k \pm 1$), hence the stationary distribution is unique.

2) Every transition changes X_n by ± 1, so it changes the parity of X_n. Hence any return to a state takes an even number of steps; since $P^2(k,k) > 0$ for all k, the period is exactly 2. In particular $P^n(0,0) = 0$ for all odd n, while (as shown next) $P^{2n}(0,0) \to 2\pi(0) > 0$; thus $P^n(0,0)$ has no limit and there is no equilibrium distribution.

   Let E and O denote the even and odd states. P^2 maps E into E, and its restriction $P^2|_E$ is an irreducible stochastic matrix on E (irreducibility of P plus parity). It is aperiodic since $P^2(0,0) = P(0,1)P(1,0) = 1/N > 0$. Since $\pi P^2 = \pi$ and P^2 does not mix E and O, the restriction $\pi|_E$ is invariant for $P^2|_E$; as $\pi(E) = 2^{-N}\sum_{k \text{ even}}\binom{N}{k} = \frac{1}{2}$, the normalized version is $2\pi|_E$. By the convergence theorem applied to $P^2|_E$,

$$\lim_{n\to\infty} P^{2n}(0,y) = \begin{cases} 2\pi(y) = 2^{1-N}\binom{N}{y}, & y \text{ even}, \\ 0, & y \text{ odd}. \end{cases}$$

   Similarly $P^{2n+1}(0,y) = \sum_z P(0,z)P^{2n}(z,y) = P^{2n}(1,y) \to 2\pi(y)$ for y odd and 0 for y even. Hence the Cesàro averages converge to $\frac{1}{2}\cdot 2\pi(y) + \frac{1}{2}\cdot 0 = \pi(y)$ for every y.

3) By Kac's formula, $m_k = 1/\pi(k) = 2^N/\binom{N}{k}$. Thus

$$m_0 = 2^N, \qquad m_{N/2} = \frac{2^N}{\binom{N}{N/2}} \sim \sqrt{\frac{\pi N}{2}} \qquad (N \to \infty),$$

   by Stirling's formula $\binom{N}{N/2} \sim 2^N\sqrt{2/(\pi N)}$. For $N = 100$, $m_0 = 2^{100} \approx 1.27 \times 10^{30}$ microseconds $\approx 1.27 \times 10^{24}$ s $\approx 4 \times 10^{16}$ years, roughly 3×10^6 times the age of the universe, whereas $m_{50} \approx 12.5$ steps.

4) Conditioning on X_n,

$$\mathbb{E}[X_{n+1} \mid X_n] = X_n + \frac{N - X_n}{N} - \frac{X_n}{N} = 1 + \left(1 - \frac{2}{N}\right)X_n.$$

   Thus $a_n = \mathbb{E}_0[X_n]$ satisfies $a_{n+1} - \frac{N}{2} = (1 - \frac{2}{N})(a_n - \frac{N}{2})$ with $a_0 = 0$, giving

$$\mathbb{E}_0[X_n] = \frac{N}{2}\left(1 - \left(1 - \frac{2}{N}\right)^n\right) \longrightarrow \frac{N}{2} = \mathbb{E}_\pi[X],$$

   since $|1 - 2/N| < 1$ for $N \ge 2$. On the other hand, with $f(x) = (-1)^x$,

$$Pf(x) = \frac{x}{N}(-1)^{x-1} + \frac{N-x}{N}(-1)^{x+1} = -(-1)^x,$$

   so $\mathbb{E}_0[(-1)^{X_n}] = (P^n f)(0) = (-1)^n$, which does not converge (and differs from $\mathbb{E}_\pi[(-1)^X] = 2^{-N}\sum_k \binom{N}{k}(-1)^k = 0$).

   Explanation. The functions $g(x) = x - \frac{N}{2}$ and $f(x) = (-1)^x$ are right eigenfunctions of P with eigenvalues $1 - \frac{2}{N}$ and -1 respectively (in fact the spectrum of P is $\{1 - \frac{2j}{N} : j = 0, \dots, N\}$, with the Krawtchouk polynomials K_j as eigenfunctions; see Conventions. Here $g = -\frac{1}{2}K_1$ and $f = K_N$). The observable x lies in the span of the constants and g, so its expectation sees only eigenvalues of modulus < 1 (other than 1) and converges. The eigenvalue -1 is the spectral signature of periodicity; it is invisible to x but detected by $(-1)^x$. Convergence of the expectations of some observables does not imply convergence in distribution.

5) $\pi Q = \frac{1}{2}(\pi + \pi P) = \pi$. Q is irreducible and $Q(x,x) = \frac{1}{2} > 0$, so Q is aperiodic; by the convergence theorem $Q^n(x,y) \to \pi(y)$, i.e. π is the equilibrium distribution of Q. Since $Qg = \frac{1}{2}(1 + 1 - \frac{2}{N})g = (1 - \frac{1}{N})g$,

$$\mathbb{E}_0^Q[X_n] = \frac{N}{2}\left(1 - \left(1 - \frac{1}{N}\right)^n\right).$$

(The spectrum of Q is $\{1 - j/N\}_{j=0}^N \subset [0,1]$, so -1 is no longer an eigenvalue; the spectral gap is $1/N$.)

REFERENCES

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf