

ECE 534 FA26 HW3 Solutions to Odd-Numbered Problems

October 6, 2026

The solutions to even-numbered problems can be found in [1, Chapter 12].

Problem 2.19 Sums of i.i.d. random variables, I

Solution. Let X_i denote the gambler's net gain for the i^{th} play. Then $X_1, X_2, \dots, X_{100}$ are i.i.d. with

$$P(X_i = x) = \begin{cases} 0.5, & x = -1, \\ 0.1, & x = 0, \\ 0.4, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$E[X_i] = -0.1, \quad \text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = 0.9 - (0.1)^2 = 0.89.$$

Let

$$S = X_1 + \dots + X_{100}.$$

Therefore

$$E[S] = -10, \quad \text{Var}(S) = (100)(0.89) = 89.$$

By the central limit theorem,

$$P(S > 0) = P\left\{\frac{S + 10}{\sqrt{89}} > \frac{10}{\sqrt{89}}\right\} \approx 1 - \Phi(1.06) \approx 0.1446.$$

Remark. It is not difficult to calculate $P(S > 0)$ numerically. The result is

$$P(S > 0) = 0.1332 \dots$$

Thus, in this example, the approximation based on the central limit theorem is fairly accurate. In fact, since S takes integer values, one can also approximate the probability of interest by

$$P(S > 0) = P(S \geq 1) = P\left(\frac{S + 10}{\sqrt{89}} \geq \frac{11}{\sqrt{89}}\right) \approx 1 - \Phi(1.166) \approx 0.1218,$$

which is also quite accurate.

Problem 2.21 Sums of i.i.d. random variables, III

Solution.

(a) A direct calculation shows that

$$\Phi_{X_{i,n}}(u) = E[e^{juX_{i,n}}] = 1 + \frac{\lambda}{n}(e^{ju} - 1).$$

Therefore, we have

$$\Phi_{Y_n}(u) = \left(1 + \frac{\lambda}{n}(e^{ju} - 1)\right)^n.$$

(b) Recall that for any $\alpha \in \mathbb{C}$ we have

$$\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha,$$

which is a direct consequence of [1, Lemma 2.14]. It follows that

$$\lim_{n \rightarrow \infty} \Phi_{Y_n}(u) = e^{\lambda(e^{ju} - 1)}.$$

This limit as a function of u is the characteristic function of a random variable Y with the Poisson distribution with mean λ .

- (c) From Part (b), we know that (Y_n) converges in distribution, and the limiting distribution is the Poisson distribution with mean λ . There is not enough information given in the problem to determine whether Y_n converges in any of the stronger senses (p., m.s., or a.s.), because the given information only describes the distribution of Y_n for each n but gives nothing about the joint distribution of the Y_n 's.

Problem 2.25 Limit behavior of a stochastic dynamical system

Solution. In the problem description, the noise variables should start at index 0 (i.e. we consider $(W_0, W_1, W_2, \dots)$ to be a sequence of i.i.d. $N(0, 0.5)$ random variables), or otherwise the recurrence relation $X_{k+1} = X_k^2 + W_k$ is ill-defined when $k = 0$.

We first show that (X_n) cannot converge in distribution to any (real-valued) random variable. In particular, we aim to show that X_n “blows up” with strictly positive probability. Define the event

$$G := \{W_0 \geq 3\} \cap \bigcap_{k=1}^{\infty} \{W_k \geq -3 \cdot 2^{k-1}\}.$$

We claim that on G we have $X_n \geq 3 \cdot 2^{n-1}$ for all $n \geq 1$, or equivalently,

$$X_n(\omega) \geq 3 \cdot 2^{n-1} \quad \text{for all } n \geq 1 \text{ and for all } \omega \in G.$$

We prove the claim by induction on n . Indeed, since $X_0(\omega) = 0$ for all $\omega \in \Omega$ by construction, we have

$$X_1(\omega) = W_0(\omega) \geq 3 \text{ for all } \omega \in G.$$

Suppose that, for some $n \geq 1$, we have for all $\omega \in G$ that

$$X_n(\omega) \geq 3 \cdot 2^{n-1}.$$

Then for all $\omega \in G$ we have

$$\begin{aligned} X_{n+1}(\omega) &= X_n^2(\omega) + W_n(\omega) \\ &\geq (3 \cdot 2^{n-1})^2 - 3 \cdot 2^{n-1} \\ &= 3 \cdot 2^{n-1} (3 \cdot 2^{n-1} - 1) \\ &\geq 3 \cdot 2^n. \end{aligned}$$

Thus the claim follows by induction.

We next show that $P(G) > 0$. By independence of $(W_0, W_1, W_2, \dots)$ and the union bound, we have

$$\begin{aligned} P(G) &= Q(3\sqrt{2})P\left(\bigcap_{k=1}^{\infty} \{W_k \geq -3 \cdot 2^{k-1}\}\right) \\ &\geq Q(3\sqrt{2})\left(1 - \sum_{k=1}^{\infty} P\{W_k < -3 \cdot 2^{k-1}\}\right). \end{aligned}$$

Since $W_k \sim N(0, 0.5)$, we have

$$P\{W_k < -3 \cdot 2^{k-1}\} = Q(3\sqrt{2} \cdot 2^{k-1}).$$

Using the inequality

$$Q(u) \leq \frac{1}{2}e^{-u^2/2}, \quad u \geq 0,^1$$

¹To see this, note that $Q(u) = \int_{x=u}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx = \exp(-u^2/2) \int_{x=u}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-(x^2 - u^2)/2) dx$, and then use the inequality $x^2 - u^2 = (x+u)(x-u) \geq (x-u)^2$ for $x \geq u \geq 0$.

we obtain

$$\begin{aligned} P(G) &\geq Q(3\sqrt{2}) \left(1 - \frac{1}{2} \sum_{k=1}^{\infty} e^{-9 \cdot 4^{k-1}} \right) \\ &\geq Q(3\sqrt{2}) \left(1 - \frac{1}{2} \sum_{k=1}^{\infty} e^{-9k} \right) \\ &= Q(3\sqrt{2}) \left(1 - \frac{e^{-9}}{2(1-e^{-9})} \right), \end{aligned}$$

where the second inequality follows from the fact that $4^{k-1} \geq k$ for all $k \geq 1$. Denote $c := Q(3\sqrt{2}) \left(1 - \frac{e^{-9}}{2(1-e^{-9})} \right)$, and we have

$$P(G) \geq c > 0.$$

Now fix any $M \in \mathbb{R}$. Note that for all n such that $3 \cdot 2^{n-1} > M$, we have on the event G that

$$X_n \geq 3 \cdot 2^{n-1} > M.$$

That is, for all n such that $3 \cdot 2^{n-1} > M$, we have

$$P(X_n > M) \geq P(G) \geq c.$$

Equivalently,

$$F_{X_n}(M) = P(X_n \leq M) \leq 1 - c$$

for all sufficiently large n .

Now assume for contradiction that

$$X_n \xrightarrow{d} X$$

for some real-valued random variable X . Then, at every continuity point M of F_X , we have

$$F_X(M) = \lim_{n \rightarrow \infty} F_{X_n}(M) \leq 1 - c.$$

Since a distribution function has at most countably many discontinuities, we may choose arbitrarily large continuity points M of F_X , and at each such point we necessarily have $F_X(M) \leq 1 - c$. This contradicts the fact that

$$\lim_{M \rightarrow \infty} F_X(M) = 1.$$

Therefore, (X_n) does not converge in distribution to any real-valued random variable.

Since convergence in distribution is implied by any one of the three other modes (a.s., p., or m.s.), it follows that the sequence (X_n) cannot converge in the a.s., p., or m.s. senses to a finite random variable either. Thus, (X_n) does not converge to any finite random variable in any of the a.s., m.s., p., or d. senses.

Remark. In fact, by adapting this proof and using Kolmogorov's zero-one law, it can be shown that X_n goes to infinity with probability one.

Problem 2.29 Mean square convergence of a random series

Solution. Recall the *Cauchy criterion for mean square convergence* [1, Proposition 2.9(b)]: A sequence (Y_n) converges in the mean-square sense if and only if both of the following hold:

- 1) $E[Y_n^2] < \infty$ for each n .
- 2) $E[(Y_m - Y_n)^2] \rightarrow 0$ as $m, n \rightarrow \infty$.

Now back to the original problem. Denote $S_n := X_1 + \cdots + X_n$ for each $n \geq 1$, and also define $S_0 := 0$ for convenience. We aim to show (S_n) converges in the mean-square sense if and only if $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$.

Suppose $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$. Then in particular for each n we have $\text{Var}(X_n) < \infty$. Then, since $(X_n)_{n \geq 1}$ are mutually independent with zero mean and finite variance, we have

$$E[S_n^2] = \text{Var}(S_n) = \sum_{i=1}^n \text{Var}(X_i) < \infty.$$

Therefore, the first condition of the Cauchy criterion for m.s. convergence is satisfied.

Then, a direct calculation shows for $m > n$ that

$$E[(S_m - S_n)^2] = E\left[\left(\sum_{i=n+1}^m X_i\right)^2\right] = \sum_{i=n+1}^m \text{Var}(X_i). \quad (1)$$

Since the partial sums $V_n = \sum_{i=1}^n \text{Var}(X_i)$ form a non-decreasing convergent sequence of real numbers, for every $\varepsilon > 0$ there exists N such that for all $m > n \geq N$ we have $V_m - V_n < \varepsilon$. It follows that for the same choice of N and for all $m > n \geq N$ we have

$$E[(S_m - S_n)^2] = \sum_{i=n+1}^m \text{Var}(X_i) = V_m - V_n < \varepsilon.$$

Hence (S_n) also satisfies the second condition of the Cauchy criterion for m.s. convergence. It follows that S_n converges in the mean-square sense.

On the other hand, suppose S_n converges in mean square to some limit S . Then by the first condition of the Cauchy criterion for m.s. convergence, we have $E[S_n^2] < \infty$ for each n . By the L^2 triangle inequality [1, (1.15)], we have for each $n \geq 1$ that

$$\begin{aligned} \sqrt{E[X_n^2]} &= \sqrt{E[(S_n - S_{n-1})^2]} \\ &\leq \sqrt{E[S_n^2]} + \sqrt{E[S_{n-1}^2]} \\ &< \infty, \end{aligned}$$

and thus $E[X_n^2] < \infty$ for each $n \geq 1$. In other words, $(X_n)_{n \geq 1}$ are independent with zero mean and finite variance, and it follows that we have the same decomposition as (1) for all $m > n$.

Now by the second condition of the Cauchy criterion for m.s. convergence, we have

$$\sum_{i=n+1}^m \text{Var}(X_i) = E[(S_m - S_n)^2] \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

This means the partial sums $V_n = \sum_{i=1}^n \text{Var}(X_i)$ form a Cauchy sequence in $\mathbb{R}$, and hence they converge. That is, we have $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$.

Combining both directions, we can deduce that $S_n = X_1 + \dots + X_n$ converges in mean square if and only if $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$.

REFERENCES

- [1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: <https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf>