Problem 1. Consider the system $\dot{x} = f(x)$, with $x \in \mathbb{R}^n$. Suppose $f(0) = 0$ and f is continuously differentiable. Furthermore, suppose there exists a $P = P^\intercal \succ 0$ such that the Jacobian $\partial f / \partial x$ satisfies:

$$P \left[\frac{\partial f}{\partial x}(x) \right] + \left[\frac{\partial f}{\partial x}(x) \right]^\intercal P \preceq -I \text{ for all } x \in \mathbb{R}^n$$

Show that:

$$x^\intercal P f(x) + f^\intercal(x) P x \preceq -x^\intercal x \text{ for all } x \in \mathbb{R}^n$$

Hint: Fix x and consider the functions $g_i(\sigma) = f_i(\sigma x)$ for each i. Each $g_i : \mathbb{R} \to \mathbb{R}$, and $f_i(x) = g_i(1)$. Consider the derivatives g_i' to find another way to represent $f(x)$.

Remark: This is known as Krasovskii's method for finding Lyapunov functions for nonlinear systems. Note that $V(x) = x^\intercal P x$ will be a positive definite and radially unbounded function which can be used to show the origin is globally exponentially stable.

Problem 2. Consider the discrete-time system:

$$x[k+1] = f(x[k])$$

We will assume that 0 is an equilibrium, i.e. $f(0) = 0$.

Consider a scalar function $V(x)$, and along trajectories of this discrete-time system, we have the following rate of change:

$$\Delta V(x) = V(f(x)) - V(x)$$

These are the analogous definitions of stability in discrete time:

1. We say the equilibrium point $x = 0$ is *stable* if: for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x[0]| < \delta$ implies $|x[k]| < \epsilon$ for all $k \geq 0$.
2. The equilibrium point is *unstable* if it is not stable.
3. The equilibrium point is *asymptotically stable* if it is stable and there exists a δ such that $|x[0]| < \delta$ implies that $\lim_{k \to \infty} x[k] = 0$.

For this problem, show the following:

a) The origin is stable if there exists a continuous function $V(x)$ such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, and $\Delta V(x) \leq 0$.

b) The origin is asymptotically stable if f is continuous and there exists a continuous function $V(x)$ such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, and $\Delta V(x) < 0$ for all $x \neq 0$.

Remark: These are the analogous Lyapunov results for discrete-time systems. Note that, since the rate of change is defined discretely, we no longer need V to be differentiable, and continuity is enough for $\Delta V(x)$ to be defined.

Let's quickly recall the proof in continuous-time. (This will more rigorously treat an error I mentioned in lecture.) Take any $\epsilon > 0$, and let $a = \min_{|x|=\epsilon} V(x) > 0$. Take any $b \in (0, a)$, and note that $\Omega_b := \{x : V(x) \leq b\}$ is an invariant set. In lecture, I incorrectly asserted that this set is contained in $\{x : |x| \leq \epsilon\}$. (This is incorrect because V can increase above b everywhere on the set $\{x : |x| = \epsilon\}$ and drop below again for some $|x| > \epsilon$.) However, we can consider the set $A = \Omega_b \cap \{x : |x| \leq \epsilon\}$. This set is contained in $\{x : |x| \leq \epsilon\}$, and we claim this intersection is in fact also invariant. Let $B = \Omega_b \cap \{x : |x| > \epsilon\}$, and note

$\Omega_b = A \cup B$. If B is empty, then $A = \Omega_b$ and we'd be done. (This is the case where the assertion in the lecture is accurate.) On the other hand, suppose B is non-empty. To show the invariance of A, it suffices to show a trajectory starting in A never enters B, since $\Omega_b = A \cup B$ is invariant. (We already know that trajectories starting in A will stay in Ω_b, at least.) Well, trajectories are continuous, so to go from A to B, the trajectory must pass through the boundary $\{x : |x| = \epsilon\}$. Since $V(x) > b$ for all x on this boundary, and $V(x(\cdot)) \leq b$ along our entire trajectory, our trajectory $x(\cdot)$ cannot go from A to B. So, A is invariant. (Essentially, the idea is that A and B are not connected, so our continuous trajectories cannot go between them. This technical detail is glossed over in the textbook.) The proof concludes by showing A contains an open ball around 0, which gives us our desired δ.

In the discrete-time case, the sublevel sets of V will be invariant, much like the continuous case. However, unlike the continuous case, the trajectories of the system are now just a sequence of discrete points. So, previous arguments about a continuous trajectory not being able to pass through a boundary are no longer valid. (Connectedness is not as important in discrete time, since a discrete-time trajectory can 'hop' across the boundary in a single time-step.) You will have to modify the proof accordingly.

Hint: In this case, setting $a = \min_{|x|=\epsilon} V(x)$ and taking any $b \in (0, a)$ will not work, for the reasons mentioned above. You'll have to find another constant b that ensures the sublevel set $\{x : V(x) \leq b\}$ is contained in B_ϵ. Continuity of V will help, and the easiest way to show this uses some tools we introduced when proving Lyapunov theorems in the time-varying case.

Problem 3. Consider the time-invariant system:

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Suppose there exists a continuously differentiable, positive definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$. Recall this implies there exists class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

Furthermore, suppose there exists a positive definite function W_3, and a positive constant $\mu > 0$ such that:

$$\frac{\partial V}{\partial x}(x) f(x) \leq -W_3(x) \text{ for all } x \text{ such that } |x| \geq \mu$$

Let $\rho = \max_{|x| \leq \mu} V(x)$.[1] Consider the set $\Omega_\rho = \{x : V(x) \leq \rho\}$. Note that $\overline{B}_\mu := \{x : |x| \leq \mu\} \subseteq \Omega_\rho$, and therefore Ω_ρ is invariant, since $\dot{V} < 0$ along its boundary.[2] Furthermore, let $r = \max_{x \in \Omega_\rho} |x| < \infty$, so $\Omega_\rho \subseteq \overline{B_r} := \{x : |x| \leq r\}$.[3]

Show the following. For any constant $c > r$, there exists a time T such that any initial condition $x(0)$ satisfying $|x(0)| \leq c$ yields:

$$|x(t)| \leq r \text{ for all } t \geq T$$

Remark: This was done in class during our discussion on non-vanishing perturbations, but many details were glossed over. You are asked here to show them in more detail. As mentioned in class, these results are known as 'ultimate boundedness', since our trajectories will 'ultimately' (i.e. eventually) be bounded.

These results are generally quite useful: they allow you to use weaker conditions (the derivative along trajectories $\dot{V}$ is only negative for larger x) and still say something about where the trajectories wind up.

Problem 4. Generalize the results from the previous problem to the time-varying case. In particular, there exists:

[1] Just another reminder that we're taking the supremum of a continuous function over a compact set, so we can write it as a maximum.

[2] To see this, note that any $|x| < \mu$ is in the interior of Ω_ρ and therefore cannot be on the boundary: there exists some $\epsilon > 0$ such that the open ball of radius ϵ centered at x is contained in Ω_ρ.

[3] And why is r finite? It doesn't have to be part of your answer, but you should know why.

- $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable
- class-$\mathcal{K}_\infty$ functions α_1 and α_2
- positive definite function W_3
- positive constant $\mu > 0$

These satisfy the following:

$$\dot{x} = f(t, x)$$

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \text{ for all } t \text{ and } x$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) \leq -W_3(x) \text{ for all } t \text{ and } x \text{ such that } |x| \geq \mu$$

Let:

$$r = \alpha_1^{-1}(\alpha_2(\mu))$$

Show the following. For any constant $c > r$, there exists a time T such that any initial time t_0 and initial condition x_0 satisfying $|x_0| \leq c$ yields:

$$|x(t)| \leq r \text{ for all } t \geq t_0 + T$$

Here, $x(t)$ is the solution starting from initial condition $x(t_0) = x_0$.
Hint: Using the $\mathcal{K}_\infty$ bounds, you'll need to find a new definition of ρ in the time-varying case. (What property did we need ρ to satisfy in the time-invariant case?) Once you find it, you can define $\Omega_{t,\rho} := \{x : V(t, x) \leq \rho\}$, which varies with t. Then, $\Omega_{t,\rho}$ is invariant, in the sense that $x(t_0) \in \Omega_{t_0,\rho}$ implies $x(t) \in \Omega_{t,\rho}$ for all $t \geq t_0$.