This homework is a review of basic notions in real analysis. As this course is more proof-heavy than previous control theory courses, we will see these methods and results used throughout the semester.

Problem 1. Recall that the topological definition of continuity: a function $f : X \to Y$ is continuous if, for every open set $U \subseteq Y$, the set $f^{-1}(U) = \{x : f(x) \in U\} \subseteq X$ is open. Additionally, recall the ϵ-δ definition of continuity: a function f is continuous if for every $x \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x' \in X$, $|x - x'| < \delta$ implies $|f(x) - f(x')| < \epsilon$.

Prove the two definitions of continuity are equivalent when the open sets are defined by the norm. (That is, a set U is open if, for every $x \in U$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) = \{x' : |x - x'| < \epsilon\} \subseteq U$.)

Hint: Draw this out and convince yourself why it's true, and then formalize that intuition afterward. Formally, all the definitions needed are provided here.

Problem 2. For a subset $A \subseteq \mathbb{R}$, the **supremum** of A is the smallest real number c such that $c \geq x$ for all $x \in A$. As such, it's often also called the **least upper bound**. If no such real number can serve as an upper bound, we say the supremum is $+\infty$. This is written as $\sup A$. (Additionally, when the set A is empty, we say $\sup A$ is $-\infty$, by convention. The reasoning is as follows: for *any* $c \in \mathbb{R}$, we have $c \geq x$ for all $x \in A$ when A is the empty set.)

Similarly, the **infimum** of A is the largest real number c such that $c \leq x$ for all $x \in A$. It is the **greatest lower bound**. You may take it for granted that every subset of $\mathbb{R}$ has a supremum and an infimum, although they could be possibly infinite.

If there exists an $x \in A$ such that $x = \sup A$, we say the supremum is **attained**. In such situations, we say $x = \max A$ as well. When the supremum is not attained, the maximum is not defined. For example, the supremum of $A = (0, 1)$ is 1, but $1 \notin A$. So, $\sup A = 1$ but the maximum is not defined.

We may also define the supremum of a function $f : X \to \mathbb{R}$ as $\sup\{f(x) : x \in X\}$. (In other words, we take the supremum over the set $A = \{f(x) : x \in X\}$.) We often write this as $\sup f$ or $\sup_x f(x)$. If there exists an $x \in A$ such that $f(x) = \sup f$, we say the supremum is **attained** and we will also write $\sup_x f(x) = \max_x f(x)$. The definition of the maximum is similar, and is undefined in the case where the supremum is not obtained.

Recall that a set is compact if every open cover has a finite subcover, and a function is continuous if the inverse image of every open set is an open set.

After all that preamble, here's the homework problem. Let $f : X \to \mathbb{R}$ be a continuous function, and suppose the domain X is compact and non-empty. Show that the supremum and infimum are attained. (Note that it suffices just to show for the supremum, as the other would follow immediately as a consequence.)

This result is known as Weierstrass's extreme value theorem, and we'll use it regularly throughout the course.

Hint: There are many ways to show this; feel free to do so however you wish. If you're stuck, here's a hint for one method. Suppose $c = \sup f$ is finite, and consider the sets $f^{-1}((-\infty, c - 1/n))$. Compactness helps greatly here. Once you finish this part of the proof, you can apply similar reasoning to the sets $f^{-1}((-\infty, n))$ to arrive at a contradiction and show that $\sup f$ must be finite.

Problem 3. Consider any norm $\|\cdot\|$ on $\mathbb{R}^n$. Prove that the unit ball $\{x : \|x\| \leq 1\}$ is convex.

Hint: As mentioned in lecture, the convexity primarily follows from the triangle inequality. However, other properties of norms are needed as well.