1 Constraint qualifications

Each item gives constraints

$$f_i(x) \le 0 \quad \forall i \in \{1, \dots, m\}, \qquad h_j(x) = 0 \quad \forall j \in \{1, \dots, p\},$$

and a feasible point x. Let $\mathcal{A}(x) = \{i : f_i(x) = 0\}$, and consider

$$\sum_{i \in \mathcal{A}(x)} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0. \tag{$*$}$$

1. **LICQ** holds at x if the only solution of $(*)$ is $y = 0$, $z = 0$.
2. **MFCQ** holds at x if the only solution of $(*)$ with $y \ge 0$ is $y = 0$, $z = 0$.

For each item, decide whether LICQ and MFCQ hold at x, and whether Slater's condition holds.

Worked example 1.1

$x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \ge 0$, at $x = (0, 0)$.

Solution.

Step 1: active set. Write $f_1 = -x_1$, $f_2 = -x_2$, and $f_3 = -x_1 - x_2$. At $x = (0, 0)$, $f_1 = f_2 = f_3 = 0$, so $\mathcal{A}(x) = \{1, 2, 3\}$. There are no equalities.

Step 2: solve $(*)$. The gradients are $\nabla f_1 = (-1, 0)$, $\nabla f_2 = (0, -1)$, and $\nabla f_3 = (-1, -1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad -y_1 - y_3 = 0, \quad -y_2 - y_3 = 0.$$

Let $y_3 = t$. The two equations give $y_1 = -t$ and $y_2 = -t$. So the solutions are $y = t(-1, -1, 1)$ for $t \in \mathbb{R}$.

Step 3: LICQ. The choice $t = 1$ gives the nonzero solution $y = (-1, -1, 1)$. So LICQ fails.

Step 4: MFCQ. A solution with $y \ge 0$ needs $y_1 = -t \ge 0$ and $y_3 = t \ge 0$. So $t = 0$, and $y = 0$ is the only solution with $y \ge 0$. So MFCQ holds.

Step 5: Slater's condition. All three constraints are affine, hence convex. At $(1, 1)$, $f_1 = -1 < 0$, $f_2 = -1 < 0$, and $f_3 = -2 < 0$. So Slater's condition holds.

Question 1

$x_1^2 + x_2^2 \le 2$, $x_1 = x_2$, at $x = (1, 1)$.

Solution

Step 1: active set. Write $f_1 = x_1^2 + x_2^2 - 2$ and $h_1 = x_1 - x_2$. At $x = (1, 1)$, $f_1 = 1 + 1 - 2 = 0$, so $\mathcal{A}(x) = \{1\}$.

Step 2: solve $(*)$. The gradients are $\nabla f_1(x) = (2x_1, 2x_2) = (2, 2)$ and $\nabla h_1 = (1, -1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + z_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad 2y_1 + z_1 = 0, \quad 2y_1 - z_1 = 0.$$

The first equation gives $z_1 = -2y_1$. Substituting into the second gives $4y_1 = 0$, so $y_1 = 0$ and $z_1 = 0$. The only solution is $y_1 = 0$, $z_1 = 0$.

Step 3: LICQ. The only solution of $(*)$ is zero. So LICQ holds.

Step 4: MFCQ. The only solution of $(*)$ is zero, so the only solution with $y \ge 0$ is zero. So MFCQ holds.

Step 5: Slater's condition. $\nabla^2 f_1 = 2I \succeq 0$, so f_1 is convex, and h_1 is affine. At $(0,0)$, $f_1 = -2 < 0$ and $h_1 = 0$. So Slater's condition holds.

Question 2

$x_2 \geq x_1^2$, $x_2 \geq 2x_1^2$, at $x = (0,0)$.

Solution

Step 1: active set. Write $f_1 = x_1^2 - x_2$ and $f_2 = 2x_1^2 - x_2$. At $x = (0,0)$, $f_1 = 0$ and $f_2 = 0$, so $\mathcal{A}(x) = \{1,2\}$.
Step 2: solve $(*)$**.** The gradients are $\nabla f_1(x) = (2x_1, -1) = (0,-1)$ and $\nabla f_2(x) = (4x_1, -1) = (0,-1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad 0 = 0, \quad -y_1 - y_2 = 0.$$

Let $y_1 = t$. The second equation gives $y_2 = -t$. So the solutions are $y = t(1,-1)$ for $t \in \mathbb{R}$.
Step 3: LICQ. The choice $t = 1$ gives the nonzero solution $y = (1,-1)$. So LICQ fails.
Step 4: MFCQ. A solution with $y \geq 0$ needs $y_1 = t \geq 0$ and $y_2 = -t \geq 0$. So $t = 0$, and $y = 0$ is the only solution with $y \geq 0$. So MFCQ holds.
Step 5: Slater's condition. $\nabla^2 f_1 = \mathrm{diag}(2,0) \succeq 0$ and $\nabla^2 f_2 = \mathrm{diag}(4,0) \succeq 0$, so both are convex. At $(0,1)$, $f_1 = -1 < 0$ and $f_2 = -1 < 0$. So Slater's condition holds.

Question 3

$x_1^2 + x_2^2 \geq 1$, at $x = (1,0)$.

Solution

Step 1: active set. Write $f_1 = 1 - x_1^2 - x_2^2$. At $x = (1,0)$, $f_1 = 1 - 1 - 0 = 0$, so $\mathcal{A}(x) = \{1\}$.
Step 2: solve $(*)$**.** The gradient is $\nabla f_1(x) = (-2x_1, -2x_2) = (-2,0)$. So $(*)$ reads

$$y_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 0, \qquad \text{that is,} \qquad -2y_1 = 0, \quad 0 = 0.$$

The first equation gives $y_1 = 0$. The only solution is $y_1 = 0$.
Step 3: LICQ. The only solution of $(*)$ is zero. So LICQ holds.
Step 4: MFCQ. The only solution of $(*)$ is zero, so the only solution with $y \geq 0$ is zero. So MFCQ holds.
Step 5: Slater's condition. $\nabla^2 f_1 = -2I$, which is not positive semidefinite. So f_1 is not convex, and Slater's condition does not apply.

Question 4

$x^2 \leq 0$, with $x \in \mathbb{R}$, at $x = 0$.

Solution

Step 1: active set. Write $f_1 = x^2$. At $x = 0$, $f_1 = 0$, so $\mathcal{A}(x) = \{1\}$.
Step 2: solve $(*)$**.** The derivative is $f_1'(x) = 2x = 0$. So $(*)$ reads $0 \cdot y_1 = 0$, and every $y_1 \in \mathbb{R}$ is a solution.

Step 3: LICQ. The choice $y_1 = 1$ is a nonzero solution. So LICQ fails.
Step 4: MFCQ. The choice $y_1 = 1$ is a nonzero solution with $y_1 \geq 0$. So MFCQ fails.
Step 5: Slater's condition. $f_1'' = 2 \geq 0$, so f_1 is convex. Slater's condition needs a point with $x^2 < 0$. Every x has $x^2 \geq 0$, so no such point exists. So Slater's condition fails.

Question 5

$x_2 \leq x_1^3$, $x_2 \geq 0$, at $x = (0, 0)$.

Solution

Step 1: active set. Write $f_1 = x_2 - x_1^3$ and $f_2 = -x_2$. At $x = (0, 0)$, $f_1 = 0$ and $f_2 = 0$, so $\mathcal{A}(x) = \{1, 2\}$.
Step 2: solve $(*)$**.** The gradients are $\nabla f_1(x) = (-3x_1^2, 1) = (0, 1)$ and $\nabla f_2 = (0, -1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad 0 = 0, \quad y_1 - y_2 = 0.$$

Let $y_1 = t$. The second equation gives $y_2 = t$. So the solutions are $y = t(1, 1)$ for $t \in \mathbb{R}$.
Step 3: LICQ. The choice $t = 1$ gives the nonzero solution $y = (1, 1)$. So LICQ fails.
Step 4: MFCQ. The choice $t = 1$ gives $y = (1, 1)$, which is nonzero and satisfies $y \geq 0$. So MFCQ fails.
Step 5: Slater's condition. $\nabla^2 f_1 = \begin{bmatrix} -6x_1 & 0 \\ 0 & 0 \end{bmatrix}$. At $x_1 = 1$ this is $\mathrm{diag}(-6, 0)$, which is not positive semidefinite. So f_1 is not convex, and Slater's condition does not apply.

Question 6

$x_1 + x_2 \leq 1$, $x_1 + x_2 \geq 1$, at $x = (1, 0)$.

Solution

Step 1: active set. Write $f_1 = x_1 + x_2 - 1$ and $f_2 = 1 - x_1 - x_2$. At $x = (1, 0)$, $f_1 = 0$ and $f_2 = 0$, so $\mathcal{A}(x) = \{1, 2\}$.
Step 2: solve $(*)$**.** The gradients are $\nabla f_1 = (1, 1)$ and $\nabla f_2 = (-1, -1)$. So $(*)$ reads

$$y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = 0, \qquad \text{that is,} \qquad y_1 - y_2 = 0, \quad y_1 - y_2 = 0.$$

Let $y_1 = t$. Then $y_2 = t$. So the solutions are $y = t(1, 1)$ for $t \in \mathbb{R}$.
Step 3: LICQ. The choice $t = 1$ gives the nonzero solution $y = (1, 1)$. So LICQ fails.
Step 4: MFCQ. The choice $t = 1$ gives $y = (1, 1)$, which is nonzero and satisfies $y \geq 0$. So MFCQ fails.
Step 5: Slater's condition. Both constraints are affine, hence convex. An affine inequality need not hold strictly in Slater's condition, so a feasible point is enough. The point $(1, 0)$ is feasible. So Slater's condition holds.
Remark. The two inequalities say $x_1 + x_2 = 1$. Written as the single equality $h_1 = x_1 + x_2 - 1 = 0$, $(*)$ reads $z_1(1, 1) = 0$, whose only solution is $z_1 = 0$, and LICQ holds.

2 Convex problems through KKT

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\}.$$

A feasible x with multipliers $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^p$ satisfies the KKT conditions if

$$\nabla f_0(x) + \sum_{i=1}^{m} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0, \qquad y \ge 0, \qquad y_i f_i(x) = 0 \ \ \forall i.$$

For each problem, complete the following steps.

1. Check Slater's condition.
2. Write the KKT conditions.
3. Solve the KKT conditions.

Worked example 2.1

$\min\ (x_1-3)^2 + x_2^2$ s.t. $x_1 \le 2$.

Solution.

Step 1: Slater's condition. Write $f_1 = x_1 - 2$. $\nabla^2 f_0 = 2I \succeq 0$, so f_0 is convex, and f_1 is affine. At $(0,0)$, $f_1 = -2 < 0$. So Slater's condition holds.

Step 2: KKT conditions. The gradients are $\nabla f_0 = (2(x_1-3), 2x_2)$ and $\nabla f_1 = (1,0)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2(x_1-3) + y_1 = 0, \quad 2x_2 = 0, \\ \text{sign:} \quad & y_1 \ge 0, \\ \text{complementary slackness:} \quad & y_1(x_1-2) = 0, \\ \text{feasibility:} \quad & x_1 - 2 \le 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases. In each case, solve the equations, then check the sign and feasibility conditions.

Case 1: $y_1 = 0$. Stationarity gives $x_1 = 3$ and $x_2 = 0$. Then $f_1 = 1 > 0$. Infeasible.

Case 2: $f_1 = 0$. Then $x_1 = 2$. Stationarity gives $y_1 = -2(2-3) = 2 \ge 0$ and $x_2 = 0$. All conditions hold.

The only KKT point is $x = (2,0)$ with $y_1 = 2$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = 1$.

Question 1

$\min\ x_1^2 + x_2^2$ s.t. $x_1 + x_2 \ge 1$.

Solution

Step 1: Slater's condition. Write $f_1 = 1 - x_1 - x_2$. $\nabla^2 f_0 = 2I \succeq 0$, so f_0 is convex, and f_1 is affine. At $(1,1)$, $f_1 = -1 < 0$. So Slater's condition holds.

Step 2: KKT conditions. The gradients are $\nabla f_0 = (2x_1, 2x_2)$ and $\nabla f_1 = (-1, -1)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2x_1 - y_1 = 0, \quad 2x_2 - y_1 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(1 - x_1 - x_2) = 0, \\ \text{feasibility:} \quad & 1 - x_1 - x_2 \leq 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. Stationarity gives $x = (0, 0)$. Then $f_1 = 1 > 0$. Infeasible.
Case 2: $f_1 = 0$. Stationarity gives $x_1 = \frac{y_1}{2}$ and $x_2 = \frac{y_1}{2}$. Substituting into $1 - x_1 - x_2 = 0$ gives $1 - y_1 = 0$, so $y_1 = 1 \geq 0$ and $x = (\frac{1}{2}, \frac{1}{2})$. All conditions hold.
The only KKT point is $x = (\frac{1}{2}, \frac{1}{2})$ with $y_1 = 1$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = \frac{1}{2}$.

Question 2

min $3x_1 + 4x_2$ s.t. $x_1^2 + x_2^2 \leq 25$.

Solution

Step 1: Slater's condition. Write $f_1 = x_1^2 + x_2^2 - 25$. f_0 is linear, hence convex. $\nabla^2 f_1 = 2I \succeq 0$, so f_1 is convex. At $(0, 0)$, $f_1 = -25 < 0$. So Slater's condition holds.
Step 2: KKT conditions. The gradients are $\nabla f_0 = (3, 4)$ and $\nabla f_1 = (2x_1, 2x_2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 3 + 2y_1 x_1 = 0, \quad 4 + 2y_1 x_2 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(x_1^2 + x_2^2 - 25) = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 25 \leq 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. The first stationarity equation reads $3 = 0$. No solution.
Case 2: $f_1 = 0$. The first stationarity equation gives $y_1 \neq 0$ and $x_1 = -\frac{3}{2y_1}$. The second gives $x_2 = -\frac{2}{y_1}$. Substituting into $x_1^2 + x_2^2 = 25$ gives $\frac{9}{4y_1^2} + \frac{4}{y_1^2} = \frac{25}{4y_1^2} = 25$, so $y_1^2 = \frac{1}{4}$ and $y_1 = \pm\frac{1}{2}$. For $y_1 = -\frac{1}{2}$, the sign condition fails. For $y_1 = \frac{1}{2}$, $x = (-3, -4)$. All conditions hold.
The only KKT point is $x = (-3, -4)$ with $y_1 = \frac{1}{2}$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = -9 - 16 = -25$.

Question 3

min $x_1^2 + x_2^2 + x_3^2$ s.t. $x_1 \geq 2$, $x_1 + x_2 + x_3 = 3$.

Solution

Step 1: Slater's condition. Write $f_1 = 2 - x_1$ and $h_1 = x_1 + x_2 + x_3 - 3$. $\nabla^2 f_0 = 2I \succeq 0$, so f_0 is convex, and f_1, h_1 are affine. At $(3, 0, 0)$, $f_1 = -1 < 0$ and $h_1 = 0$. So Slater's condition holds.

Step 2: KKT conditions. The gradients are $\nabla f_0 = (2x_1, 2x_2, 2x_3)$, $\nabla f_1 = (-1, 0, 0)$, and $\nabla h_1 = (1, 1, 1)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2x_1 - y_1 + z_1 = 0, \quad 2x_2 + z_1 = 0, \quad 2x_3 + z_1 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(2 - x_1) = 0, \\ \text{feasibility:} \quad & 2 - x_1 \leq 0, \quad x_1 + x_2 + x_3 - 3 = 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. Stationarity gives $x_1 = x_2 = x_3 = -\frac{z_1}{2}$. Substituting into $h_1 = 0$ gives $-\frac{3z_1}{2} = 3$, so $z_1 = -2$ and $x = (1, 1, 1)$. Then $f_1 = 1 > 0$. Infeasible.
Case 2: $f_1 = 0$. Then $x_1 = 2$. The second and third stationarity equations give $x_2 = x_3 = -\frac{z_1}{2}$. Substituting into $h_1 = 0$ gives $2 - z_1 - 3 = 0$, so $z_1 = -1$ and $x_2 = x_3 = \frac{1}{2}$. The first stationarity equation gives $y_1 = 2x_1 + z_1 = 4 - 1 = 3 \geq 0$. All conditions hold.
The only KKT point is $x = (2, \frac{1}{2}, \frac{1}{2})$ with $y_1 = 3$ and $z_1 = -1$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = \frac{9}{2}$.

Question 4

min $-\log x_1 - \log x_2$ s.t. $x_1 + 2x_2 \leq 4$, over the domain $x > 0$.

Solution

Step 1: Slater's condition. Write $f_1 = x_1 + 2x_2 - 4$. On the domain, $\nabla^2 f_0 = \operatorname{diag}(\frac{1}{x_1^2}, \frac{1}{x_2^2}) \succeq 0$, so f_0 is convex, and f_1 is affine. The point $(1, 1)$ lies in the domain and gives $f_1 = -1 < 0$. So Slater's condition holds.
Step 2: KKT conditions. The gradients are $\nabla f_0 = (-\frac{1}{x_1}, -\frac{1}{x_2})$ and $\nabla f_1 = (1, 2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & -\frac{1}{x_1} + y_1 = 0, \quad -\frac{1}{x_2} + 2y_1 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(x_1 + 2x_2 - 4) = 0, \\ \text{feasibility:} \quad & x_1 + 2x_2 - 4 \leq 0, \quad x > 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. The first stationarity equation reads $\frac{1}{x_1} = 0$. No solution.
Case 2: $f_1 = 0$. The first stationarity equation gives $y_1 = \frac{1}{x_1} \neq 0$, so $x_1 = \frac{1}{y_1}$. The second gives $x_2 = \frac{1}{2y_1}$. Substituting into $x_1 + 2x_2 - 4 = 0$ gives $\frac{2}{y_1} = 4$, so $y_1 = \frac{1}{2} \geq 0$ and $x = (2, 1)$. All conditions hold.
The only KKT point is $x = (2, 1)$ with $y_1 = \frac{1}{2}$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = -\log 2$.

Question 5

(G) min $(x_1 - 2)^2 + (x_2 - 2)^2$ s.t. $x_1^2 + x_2^2 \leq 2$.

Solution

Step 1: Slater's condition. Write $f_1 = x_1^2 + x_2^2 - 2$. $\nabla^2 f_0 = 2I \succeq 0$ and $\nabla^2 f_1 = 2I \succeq 0$, so both are convex. At $(0,0)$, $f_1 = -2 < 0$. So Slater's condition holds.
Step 2: KKT conditions. The gradients are $\nabla f_0 = (2(x_1-2), 2(x_2-2))$ and $\nabla f_1 = (2x_1, 2x_2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2(x_1-2) + 2y_1 x_1 = 0, \quad 2(x_2-2) + 2y_1 x_2 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(x_1^2 + x_2^2 - 2) = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 2 \leq 0. \end{aligned}$$

Step 3: solve. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. Stationarity gives $x = (2,2)$. Then $f_1 = 6 > 0$. Infeasible.
Case 2: $f_1 = 0$. Stationarity gives $(1+y_1)x_1 = 2$ and $(1+y_1)x_2 = 2$. So $1 + y_1 \neq 0$ and $x_1 = x_2 = \frac{2}{1+y_1}$. Substituting into $x_1^2 + x_2^2 = 2$ gives $\frac{8}{(1+y_1)^2} = 2$, so $(1+y_1)^2 = 4$ and $y_1 = 1$ or $y_1 = -3$. For $y_1 = -3$, the sign condition fails. For $y_1 = 1$, $x = (1,1)$. All conditions hold.
The only KKT point is $x = (1,1)$ with $y_1 = 1$. The problem is convex and Slater's condition holds, so x is a global minimizer, with $f_0(x) = 1 + 1 = 2$.

3 Nonconvex problems through KKT

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\}.$$

A feasible x with multipliers $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^p$ satisfies the KKT conditions if

$$\nabla f_0(x) + \sum_{i=1}^{m} y_i \nabla f_i(x) + \sum_{j=1}^{p} z_j \nabla h_j(x) = 0, \qquad y \ge 0, \qquad y_i f_i(x) = 0 \ \ \forall i.$$

In each problem below, the feasible set is compact and LICQ holds at every feasible point. For each problem, complete the following steps.

1. Write the KKT conditions.
2. Find all KKT points.
3. Find the global minimizer.

Worked example 3.1

min $x_1 x_2$ s.t. $x_1^2 + x_2^2 \le 2$.

Solution.

Step 1: KKT conditions. Write $f_1 = x_1^2 + x_2^2 - 2$. The gradients are $\nabla f_0 = (x_2, x_1)$ and $\nabla f_1 = (2x_1, 2x_2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & x_2 + 2y_1 x_1 = 0, \quad x_1 + 2y_1 x_2 = 0, \\ \text{sign:} \quad & y_1 \ge 0, \\ \text{complementary slackness:} \quad & y_1(x_1^2 + x_2^2 - 2) = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 2 \le 0. \end{aligned}$$

Step 2: KKT points. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.

Case 1: $y_1 = 0$. Stationarity gives $x_2 = 0$ and $x_1 = 0$. Then $f_1 = -2 \le 0$. So $x = (0,0)$ with $y_1 = 0$ is a KKT point.

Case 2: $f_1 = 0$. The first stationarity equation gives $x_2 = -2y_1 x_1$. Substituting into the second gives $x_1 - 4y_1^2 x_1 = 0$, that is, $x_1(1 - 4y_1^2) = 0$. If $x_1 = 0$, then $x_2 = 0$, which contradicts $x_1^2 + x_2^2 = 2$. So $1 - 4y_1^2 = 0$ and $y_1 = \pm\frac{1}{2}$. For $y_1 = -\frac{1}{2}$, the sign condition fails. For $y_1 = \frac{1}{2}$, $x_2 = -x_1$, and $x_1^2 + x_2^2 = 2$ gives $2x_1^2 = 2$, so $x_1 = \pm 1$. So $x = (1,-1)$ and $x = (-1,1)$ with $y_1 = \frac{1}{2}$ are KKT points.

Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(0,0) = 0, \qquad f_0(1,-1) = -1, \qquad f_0(-1,1) = -1.$$

So the global minimizers are $(1,-1)$ and $(-1,1)$, with $p^\star = -1$.

Question 1

min $x_1^2 - x_2^2$ s.t. $x_1^2 + x_2^2 \le 1$.

Solution

Step 1: KKT conditions. Write $f_1 = x_1^2 + x_2^2 - 1$. The gradients are $\nabla f_0 = (2x_1, -2x_2)$ and $\nabla f_1 = (2x_1, 2x_2)$. The KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2x_1 + 2y_1 x_1 = 0, \quad -2x_2 + 2y_1 x_2 = 0, \\ \text{sign:} \quad & y_1 \geq 0, \\ \text{complementary slackness:} \quad & y_1(x_1^2 + x_2^2 - 1) = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 1 \leq 0. \end{aligned}$$

Step 2: KKT points. Complementary slackness requires $y_1 = 0$ or $f_1(x) = 0$. This gives two cases.
Case 1: $y_1 = 0$. Stationarity gives $2x_1 = 0$ and $-2x_2 = 0$, so $x = (0, 0)$. Then $f_1 = -1 \leq 0$. So $x = (0, 0)$ with $y_1 = 0$ is a KKT point.
Case 2: $f_1 = 0$. The first stationarity equation reads $2x_1(1 + y_1) = 0$, so $x_1 = 0$ or $y_1 = -1$. If $y_1 = -1$, the sign condition fails. If $x_1 = 0$, then $x_1^2 + x_2^2 = 1$ gives $x_2 = \pm 1$. The second stationarity equation reads $2x_2(y_1 - 1) = 0$, and $x_2 \neq 0$, so $y_1 = 1 \geq 0$. So $x = (0, 1)$ and $x = (0, -1)$ with $y_1 = 1$ are KKT points.
Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(0, 0) = 0, \qquad f_0(0, 1) = -1, \qquad f_0(0, -1) = -1.$$

So the global minimizers are $(0, 1)$ and $(0, -1)$, with $p^\star = -1$.

Question 2

min $x_1 + 2x_2$ s.t. $x_1^2 + x_2^2 = 5$.

Solution

Step 1: KKT conditions. Write $h_1 = x_1^2 + x_2^2 - 5$. The gradients are $\nabla f_0 = (1, 2)$ and $\nabla h_1 = (2x_1, 2x_2)$. There are no inequalities, so the KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 1 + 2z_1 x_1 = 0, \quad 2 + 2z_1 x_2 = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 5 = 0. \end{aligned}$$

Step 2: KKT points. The first stationarity equation gives $z_1 \neq 0$ and $x_1 = -\frac{1}{2z_1}$. The second gives $x_2 = -\frac{1}{z_1}$. Substituting into $x_1^2 + x_2^2 = 5$ gives $\frac{1}{4z_1^2} + \frac{1}{z_1^2} = \frac{5}{4z_1^2} = 5$, so $z_1^2 = \frac{1}{4}$ and $z_1 = \pm\frac{1}{2}$. A multiplier of an equality has no sign condition. For $z_1 = \frac{1}{2}$, $x = (-1, -2)$. For $z_1 = -\frac{1}{2}$, $x = (1, 2)$. Both are KKT points.
Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(-1, -2) = -5, \qquad f_0(1, 2) = 5.$$

So the global minimizer is $(-1, -2)$, with $p^\star = -5$.

Question 3

min $x_1^2 + 2x_2^2$ s.t. $x_1^2 + x_2^2 = 1$.

Solution

Step 1: KKT conditions. Write $h_1 = x_1^2 + x_2^2 - 1$. The gradients are $\nabla f_0 = (2x_1, 4x_2)$ and $\nabla h_1 = (2x_1, 2x_2)$. There are no inequalities, so the KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 2x_1 + 2z_1 x_1 = 0, \quad 4x_2 + 2z_1 x_2 = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 - 1 = 0. \end{aligned}$$

Step 2: KKT points. The stationarity equations read $2x_1(1 + z_1) = 0$ and $2x_2(2 + z_1) = 0$.
Case 1: $z_1 = -1$. The second equation reads $2x_2 = 0$, so $x_2 = 0$. Feasibility gives $x_1^2 = 1$, so $x_1 = \pm 1$. So $x = (1, 0)$ and $x = (-1, 0)$ with $z_1 = -1$ are KKT points.
Case 2: $z_1 \neq -1$. The first equation gives $x_1 = 0$. Feasibility gives $x_2^2 = 1$, so $x_2 = \pm 1 \neq 0$. The second equation then gives $z_1 = -2$. So $x = (0, 1)$ and $x = (0, -1)$ with $z_1 = -2$ are KKT points.
Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(\pm 1, 0) = 1, \qquad f_0(0, \pm 1) = 2.$$

So the global minimizers are $(1, 0)$ and $(-1, 0)$, with $p^\star = 1$.

Question 4

(G) min $x_1 + 2x_2 + 2x_3$ s.t. $x_1^2 + x_2^2 + x_3^2 = 9$.

Solution

Step 1: KKT conditions. Write $h_1 = x_1^2 + x_2^2 + x_3^2 - 9$. The gradients are $\nabla f_0 = (1, 2, 2)$ and $\nabla h_1 = (2x_1, 2x_2, 2x_3)$. There are no inequalities, so the KKT conditions are

$$\begin{aligned} \text{stationarity:} \quad & 1 + 2z_1 x_1 = 0, \quad 2 + 2z_1 x_2 = 0, \quad 2 + 2z_1 x_3 = 0, \\ \text{feasibility:} \quad & x_1^2 + x_2^2 + x_3^2 - 9 = 0. \end{aligned}$$

Step 2: KKT points. The first stationarity equation gives $z_1 \neq 0$ and $x_1 = -\frac{1}{2z_1}$. The second and third give $x_2 = -\frac{1}{z_1}$ and $x_3 = -\frac{1}{z_1}$. Substituting into $x_1^2 + x_2^2 + x_3^2 = 9$ gives $\frac{1}{4z_1^2} + \frac{1}{z_1^2} + \frac{1}{z_1^2} = \frac{9}{4z_1^2} = 9$, so $z_1^2 = \frac{1}{4}$ and $z_1 = \pm\frac{1}{2}$. A multiplier of an equality has no sign condition. For $z_1 = \frac{1}{2}$, $x = (-1, -2, -2)$. For $z_1 = -\frac{1}{2}$, $x = (1, 2, 2)$. Both are KKT points.
Step 3: global minimizer. The feasible set is compact, so a global minimizer exists. LICQ holds at every feasible point, so the global minimizer is one of the KKT points. The values are

$$f_0(-1, -2, -2) = -9, \qquad f_0(1, 2, 2) = 9.$$

So the global minimizer is $(-1, -2, -2)$, with $p^\star = -9$.

4 Second-order conditions

Consider

$$\min_x \ f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \ \ \forall i \in \{1,\dots,m\}, \qquad h_j(x) = 0 \ \ \forall j \in \{1,\dots,p\},$$

with Lagrangian

$$\mathscr{L}(x,y,z) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x) + \sum_{j=1}^{p} z_j h_j(x).$$

At a KKT point $(\hat{x}, y, z)$, the critical cone is

$$\mathcal{C}(\hat{x},y) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \nabla h_j(\hat{x})^T d = 0 & \forall j, \\ \nabla f_i(\hat{x})^T d = 0 & \forall i \text{ with } y_i > 0, \\ \nabla f_i(\hat{x})^T d \le 0 & \forall i \text{ with } f_i(\hat{x}) = 0 \text{ and } y_i = 0 \end{array} \right\}.$$

Each item gives a KKT point $\hat{x}$ with its multipliers, and LICQ holds at $\hat{x}$. For each item, complete the following steps.

1. Write the critical cone explicitly in terms of $d_1, \dots, d_n$.
2. Check the second-order necessary condition.
3. Check the second-order sufficient condition.

Worked example 4.1

min $x_1 + x_2$ s.t. $x_1^2 + x_2^2 \le 2$, $x_1 \le -1$, at $\hat{x} = (-1,-1)$ with $y = (\frac{1}{2}, 0)$.

Solution.

Step 1: critical cone. Write $f_1 = x_1^2 + x_2^2 - 2$ and $f_2 = x_1 + 1$. At $\hat{x}$, $f_1 = 0$ and $f_2 = 0$, so both are active. The gradients are $\nabla f_1(\hat{x}) = (2x_1, 2x_2) = (-2,-2)$ and $\nabla f_2 = (1,0)$. Since $y_1 = \frac{1}{2} > 0$, constraint 1 gives an equation. Since $y_2 = 0$, constraint 2 gives an inequality. So

$$\mathcal{C}(\hat{x},y) = \{d : -2d_1 - 2d_2 = 0, \ d_1 \le 0\}.$$

The equation gives $d_2 = -d_1$. Let $d_1 = -t$. The inequality $d_1 \le 0$ becomes $t \ge 0$. So $\mathcal{C}(\hat{x},y) = \{(-t,t) : t \ge 0\}$.

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = 0$, $\nabla^2 f_1 = 2I$, and $\nabla^2 f_2 = 0$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x},y,z) = \nabla^2 f_0 + y_1 \nabla^2 f_1 + y_2 \nabla^2 f_2 = 0 + \tfrac{1}{2}(2I) + 0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For $d = (-t, t)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = d_1^2 + d_2^2 = (-t)^2 + t^2 = 2t^2 \ge 0.$$

So the necessary condition holds.

Step 3: sufficient condition. A nonzero $d \in \mathcal{C}(\hat{x},y)$ has $t > 0$, so $d^T \nabla^2_{xx}\mathscr{L}\, d = 2t^2 > 0$. So the sufficient condition holds, and $\hat{x}$ is a strict local minimizer.

Question 1

(G) min $x_1 + x_2$ s.t. $x_1 x_2 = 1$, at $\hat{x} = (-1,-1)$ with $z = 1$.

Solution

Step 1: critical cone. Write $h_1 = x_1x_2 - 1$. There are no inequalities. The gradient is $\nabla h_1(\hat{x}) = (x_2, x_1) = (-1, -1)$. So

$$\mathcal{C}(\hat{x}, y) = \{d : -d_1 - d_2 = 0\}.$$

The equation gives $d_2 = -d_1$. Let $d_1 = t$. So $\mathcal{C}(\hat{x}, y) = \{(t, -t) : t \in \mathbb{R}\}$.

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = 0$ and $\nabla^2 h_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x}, y, z) = \nabla^2 f_0 + z_1 \nabla^2 h_1 = 0 + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $d = (t, -t)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = 2d_1d_2 = 2(t)(-t) = -2t^2.$$

For $t = 1$, this is $-2 < 0$. So the necessary condition fails. Since LICQ holds, $\hat{x}$ is not a local minimizer.

Step 3: sufficient condition. The nonzero direction $d = (1, -1) \in \mathcal{C}(\hat{x}, y)$ gives $d^T \nabla^2_{xx}\mathscr{L}\, d = -2$, which is not > 0. So the sufficient condition fails.

Question 2

(G) min $x_1^2 + x_2^2$ s.t. $x_1^2 + \frac{1}{4}x_2^2 \geq 1$, at $\hat{x} = (0, 2)$ with $y = 4$.

Solution

Step 1: critical cone. Write $f_1 = 1 - x_1^2 - \frac{1}{4}x_2^2$. At $\hat{x}$, $f_1 = 1 - 0 - 1 = 0$, so it is active. The gradient is $\nabla f_1(\hat{x}) = (-2x_1, -\frac{1}{2}x_2) = (0, -1)$. Since $y_1 = 4 > 0$, constraint 1 gives an equation. So

$$\mathcal{C}(\hat{x}, y) = \{d : -d_2 = 0\}.$$

The equation gives $d_2 = 0$. Let $d_1 = t$. So $\mathcal{C}(\hat{x}, y) = \{(t, 0) : t \in \mathbb{R}\}$.

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = \operatorname{diag}(2, 2)$ and $\nabla^2 f_1 = \operatorname{diag}(-2, -\frac{1}{2})$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x}, y, z) = \nabla^2 f_0 + y_1 \nabla^2 f_1 = \operatorname{diag}(2, 2) + 4\operatorname{diag}(-2, -\tfrac{1}{2}) = \operatorname{diag}(-6, 0).$$

For $d = (t, 0)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = -6d_1^2 + 0 \cdot d_2^2 = -6t^2.$$

For $t = 1$, this is $-6 < 0$. So the necessary condition fails. Since LICQ holds, $\hat{x}$ is not a local minimizer.

Step 3: sufficient condition. The nonzero direction $d = (1, 0) \in \mathcal{C}(\hat{x}, y)$ gives $d^T \nabla^2_{xx}\mathscr{L}\, d = -6$, which is not > 0. So the sufficient condition fails.

Question 3

(G) min $x_1^2 + 4x_1x_2 + x_2^2$ s.t. $x_1 \geq 0$, $x_2 \geq 0$, at $\hat{x} = (0, 0)$ with $y = (0, 0)$.

Solution

Step 1: critical cone. Write $f_1 = -x_1$ and $f_2 = -x_2$. At $\hat{x}$, $f_1 = 0$ and $f_2 = 0$, so both are active. The gradients are $\nabla f_1 = (-1, 0)$ and $\nabla f_2 = (0, -1)$. Since $y_1 = 0$ and $y_2 = 0$, both

constraints give inequalities. So

$$\mathcal{C}(\hat{x}, y) = \{d : -d_1 \le 0,\ -d_2 \le 0\} = \{d : d_1 \ge 0,\ d_2 \ge 0\}.$$

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ and $\nabla^2 f_1 = \nabla^2 f_2 = 0$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x}, y, z) = \nabla^2 f_0 + y_1 \nabla^2 f_1 + y_2 \nabla^2 f_2 = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}.$$

For $d = (d_1, d_2)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = 2d_1^2 + 8d_1 d_2 + 2d_2^2.$$

For $d \in \mathcal{C}(\hat{x}, y)$, $d_1 \ge 0$ and $d_2 \ge 0$, so each of the three terms is ≥ 0, and the sum is ≥ 0. So the necessary condition holds.

Step 3: sufficient condition. Let $d \in \mathcal{C}(\hat{x}, y)$ be nonzero. Then $d_1 > 0$ or $d_2 > 0$. If $d_1 > 0$, then $2d_1^2 > 0$, and the other two terms are ≥ 0, so the sum is > 0. If $d_2 > 0$, then $2d_2^2 > 0$, and the other two terms are ≥ 0, so the sum is > 0. So the sufficient condition holds, and $\hat{x}$ is a strict local minimizer.

Question 4

(G) min $x_1 + x_2^2$ s.t. $x_2 \ge 0$, $x_1^2 + x_2^2 = 1$, at $\hat{x} = (1, 0)$ with $y = 0$ and $z = -\frac{1}{2}$.

Solution

Step 1: critical cone. Write $f_1 = -x_2$ and $h_1 = x_1^2 + x_2^2 - 1$. At $\hat{x}$, $f_1 = 0$, so it is active. The gradients are $\nabla f_1 = (0, -1)$ and $\nabla h_1(\hat{x}) = (2x_1, 2x_2) = (2, 0)$. The equality gives an equation. Since $y_1 = 0$, constraint 1 gives an inequality. So

$$\mathcal{C}(\hat{x}, y) = \{d : 2d_1 = 0,\ -d_2 \le 0\}.$$

The equation gives $d_1 = 0$. Let $d_2 = t$. The inequality $-d_2 \le 0$ becomes $t \ge 0$. So $\mathcal{C}(\hat{x}, y) = \{(0, t) : t \ge 0\}$.

Step 2: necessary condition. The Hessians are $\nabla^2 f_0 = \mathrm{diag}(0, 2)$, $\nabla^2 f_1 = 0$, and $\nabla^2 h_1 = \mathrm{diag}(2, 2)$. So

$$\nabla^2_{xx}\mathscr{L}(\hat{x}, y, z) = \nabla^2 f_0 + y_1 \nabla^2 f_1 + z_1 \nabla^2 h_1 = \mathrm{diag}(0, 2) + 0 - \tfrac{1}{2}\,\mathrm{diag}(2, 2) = \mathrm{diag}(-1, 1).$$

For $d = (0, t)$,

$$d^T \nabla^2_{xx}\mathscr{L}\, d = -d_1^2 + d_2^2 = 0 + t^2 = t^2 \ge 0.$$

So the necessary condition holds.

Step 3: sufficient condition. A nonzero $d \in \mathcal{C}(\hat{x}, y)$ has $t > 0$, so $d^T \nabla^2_{xx}\mathscr{L}\, d = t^2 > 0$. So the sufficient condition holds, and $\hat{x}$ is a strict local minimizer.