UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 9 Solutions

Modeling & full-state feedback:

Solutions

Recall the Controllable Canonical Form: Starting with

$$(s^3 + a_2 s^2 + a_1 s + a_0)Y(s) = (b_0 + b_1 s + b_2 s^2)U(s)$$

we obtain the state space model,

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$
$$y = (b_0 \;\; b_1 \;\; b_2 \;)x + (0)u.$$

In each problem we can take this form for simplicity.
Also, recall that with state feedback $u = -Kx + r$, the closed loop system is also in CCF:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 - k_1 & -a_1 - k_2 & -a_2 - k_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r$$
$$y = (b_0 \;\; b_1 \;\; b_2 \;)x$$

Problem 1

Consider the SISO model,

$$Y(s) = \frac{s+1}{s^2 + 2s + 2} U(s)$$

A second-order state-space model (without Matlab):

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = (1 \;\; 1)x$$

If we want to place the closed loop poles at -4 and -25, then the new denominator would be $(s+4)(s+25) = s^2 + 29s + 100$. Therefore, $a_0 + k_1 = 100 \Rightarrow k_1 = 100 - 2 = 98$ and

$a_1 + k_2 = 29 \Rightarrow k_2 = 29 - 2 = 27$
Hence, $K = [98 \quad 27] \Rightarrow u = -[98 \quad 27]x + r$

Problem 2

Consider the satellite position model with delay,

$$G_p(s) = \frac{1 - \frac{1}{2}s}{1 + \frac{1}{2}s} \cdot \frac{1}{s^2}$$
$$\Rightarrow (2s^2 + s^3)Y(s) = (2 - s)U(s)$$

A third-order state-space model (without Matlab):

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$
$$y = (2 \quad -1 \quad 0\ )x$$

If we want to place the closed loop poles at -4, -13 and -25, then the new denominator would be $(s+4)(s+13)(s+25) = s^3 + 42s^2 + 477s + 1300$.

Follow the step that we did earlier in Problem 1, we will find that $a_0 + k_1 = 1300 \Rightarrow k_1 = 1300 - 0 = 1300$, $a_1 + k_2 = 477 \Rightarrow k_2 = 477 - 0 = 477$, and $a_2 + k_3 = 42 \Rightarrow k_3 = 42 - 2 = 40$

Hence, $K = [1300 \quad 477 \quad 40] \Rightarrow u = -[1300 \quad 477 \quad 40]x + r$

Observers & sensitivity:

Solutions

For an observer gain L, the observer poles are the eigenvalues of $A-LC$, which coincide with the eigenvalues of $A' - C'L'$, where the "prime" denotes transpose. Consequently, to compute the observer gain in Matlab we apply the `place` command for (A', C').

Problem 3

Return to the feedback system considered in Problem 1:

(a) Construct a stable observer to estimate x based on measurements of (u, y).
It is a rule of thumb to pick observer poles to be 2 - 5 times further than the controller poles
In Problem 1, the controller poles are at -4 and -25. Suppose we want to place the observer poles at {-50 -51}. By using a Matlab command

$$\mathtt{L} = \mathtt{place}(\mathtt{A}', \mathtt{C}', [-50 - 51])$$

where A' and C' are from Problem 1. We find that, $L = \begin{pmatrix} -2449 \\ 2548 \end{pmatrix}$

(b) Obtain a state-feedback compensator $u = -K\hat{x} + k_r r$, where K was obtained in your prior work, and k_r is chosen so that the DC gain Y/R is equal to unity.

To obtain k_r, recall that $\tilde{x}(t) \equiv 0$ if $\tilde{x}(0) = 0$, so we can ignore the observer – The closed loop system transfer function disregards initial conditions. With full state feedback, the closed loop transfer function is $Y(s)/R(s) = C[Is-(A-BK)]^{-1}Bk_r$. To set the DC gain to unity we need,

$$1 = C[-(A - BK)]^{-1}Bk_r$$
$$\Rightarrow k_r = 1/(C[-(A - BK)]^{-1}B) = 100$$

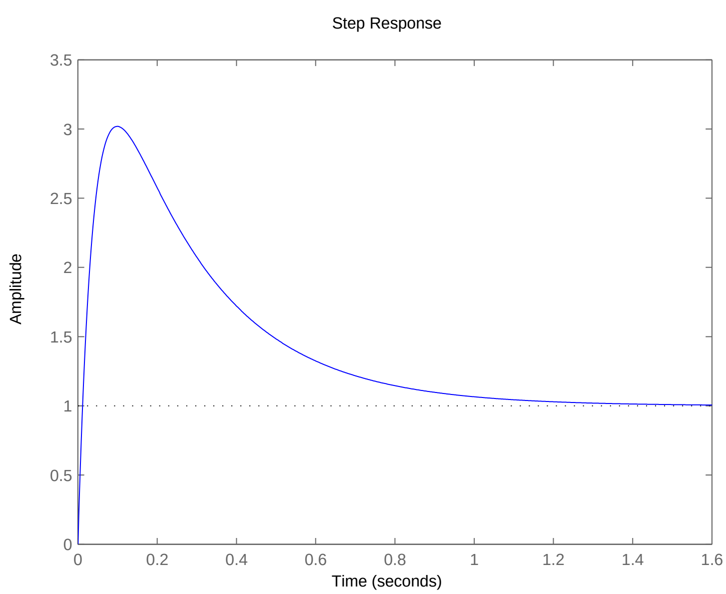
(c) Obtain a step response using Matlab.

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - LC \end{array}\right]}_{A_{cl}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \underbrace{\left[\begin{array}{c} k_r B \\ \hline 0 \end{array}\right]}_{B_{cl}} r$$

$$y = \underbrace{[C \mid 0]}_{C_{cl}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

In Matlab:

```
Acl =

           0           1           0           0
        -100         -29          98          27
           0           0        2449        2450
           0           0       -2550       -2550

Bcl =

     0
   100
     0
     0


Ccl =

     1     1     0     0
```



Overshoot is as expected due to the LHP zero.

Problem 4

Return to the feedback system considered in Problem 2:

(a) Construct a stable observer, and using this obtain a compensator of the form $U = -G_c Y + G_r R$.

Suppose we want to place the observer poles at {-50 -51 -52}. By using a Matlab, we found that

```
L2 =
    9301
    18450
    29400
```

The feedback law $u = -K\hat{x} + k_r r$ can be expressed in the frequency domain (subject to zero initial conditions) by $U = -G_c Y + G_r R$, with

$$\begin{aligned} G_c(s) &= K[Is - (A - BK - LC)]^{-1} L \\ G_r(s) &= I - K[Is - (A - BK - LC)]^{-1} B \end{aligned}$$

We can obtain $G_c(s)$ from Matlab:

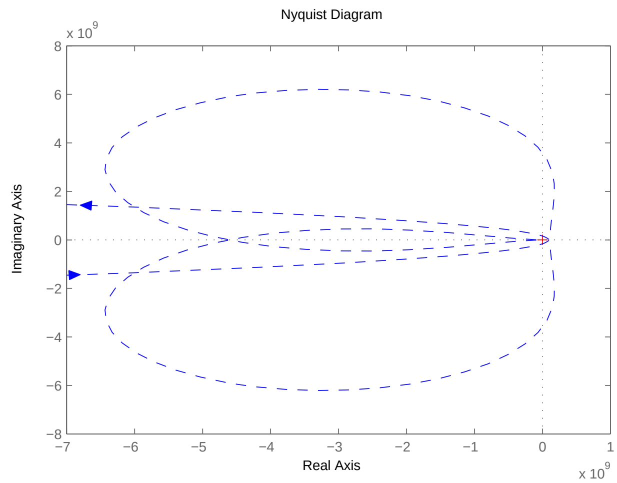
```
Acl2 =

  1.0e+004 *

   -1.8601    0.9302         0
   -3.6900    1.8450    0.0001
   -6.0100    2.8923   -0.0042

G_c(s) =
Transfer function:
2.207e007 s^2 + 7.979e007 s + 8.619e007
---------------------------------------
s^3 + 193 s^2 + 1.432e004 s + 2.257e007
```

The resulting transfer function G_c has zeros at $-1.8079 \pm 0.7983j$, and poles at

```
  1.0e+002 *
  -3.4306
   0.7503 + 2.4530i
   0.7503 - 2.4530i
```

Note that introduces two poles in the RHP.

(b) Obtain a Nyquist plot for $G_c G_p$ using Matlab, and estimate the gain and phase margins.



We have $P = 2$ (from the compensator), and we do see that $N = -2$, given $Z = 0$. The system is stable, but there is virtually no gain or phase margin.