UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 7 Solutions

Spring 2024

Problem 1

Consider the transfer function

$$H(s) = \frac{1}{s+a},$$

where $a > 0$. Prove that the Nyquist plot of H is a circle of radius $\frac{1}{2a}$ centered at the point $\left(\frac{1}{2a}, 0\right)$.

For any ω, we have

$$\begin{aligned} H(j\omega) &= \frac{1}{j\omega + a} \\ &= \frac{-j\omega + a}{(j\omega + a)(-j\omega + a)} \\ &= \frac{a - j\omega}{a^2 + \omega^2}. \end{aligned}$$

Therefore, the Nyquist plot of H has the parametric form

$$\Big(\operatorname{Re} H(j\omega), \operatorname{Im} H(j\omega)\Big) = \left(\frac{a}{a^2+\omega^2}, -\frac{\omega}{a^2+\omega^2}\right), \qquad -\infty < \omega < \infty.$$

Recall that the points (x, y) that lie on a circle of radius r centered at the point (a, b) satisfy

$$(x-a)^2 + (y-b)^2 = r^2.$$

Thus, we compute

$$
\begin{aligned}
&\left(\operatorname{Re} H(j\omega) - \frac{1}{2a}\right)^2 + (\operatorname{Im} H(j\omega))^2 \\
&\quad = \left(\frac{a}{a^2+\omega^2} - \frac{a^2+\omega^2}{2a(a^2+\omega^2)}\right)^2 + \left(\frac{\omega}{a^2+\omega^2}\right)^2 \\
&\quad = \left(\frac{a^2-\omega^2}{2a(a^2+\omega^2)}\right)^2 + \left(\frac{2a\omega}{2a(a^2+\omega^2)}\right)^2 \\
&\quad = \left(\frac{1}{2a}\right)^2 \left[(a^2-\omega^2)^2 + 4a^2\omega^2\right] \\
&\quad = \left(\frac{1}{2a}\right)^2 \frac{a^4 - 2a^2\omega^2 + \omega^4 + 4a^2\omega^2}{(a^2+\omega^2)^2} \\
&\quad = \left(\frac{1}{2a}\right)^2 \frac{a^4 + 2a^2\omega^2 + \omega^4}{(a^2+\omega^2)^2} \\
&\quad = \left(\frac{1}{2a}\right)^2.
\end{aligned}
$$

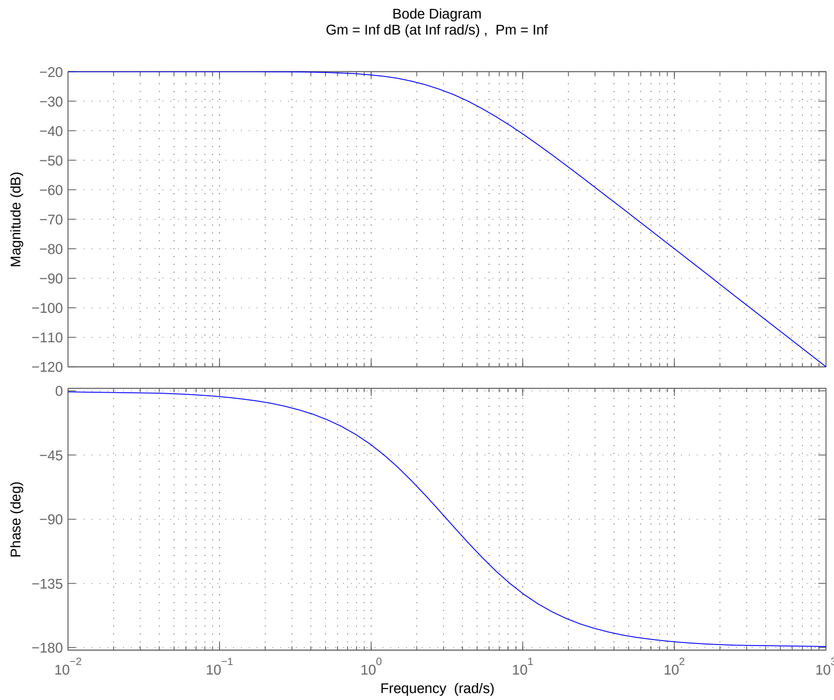
This proves the claim.

Problem 2

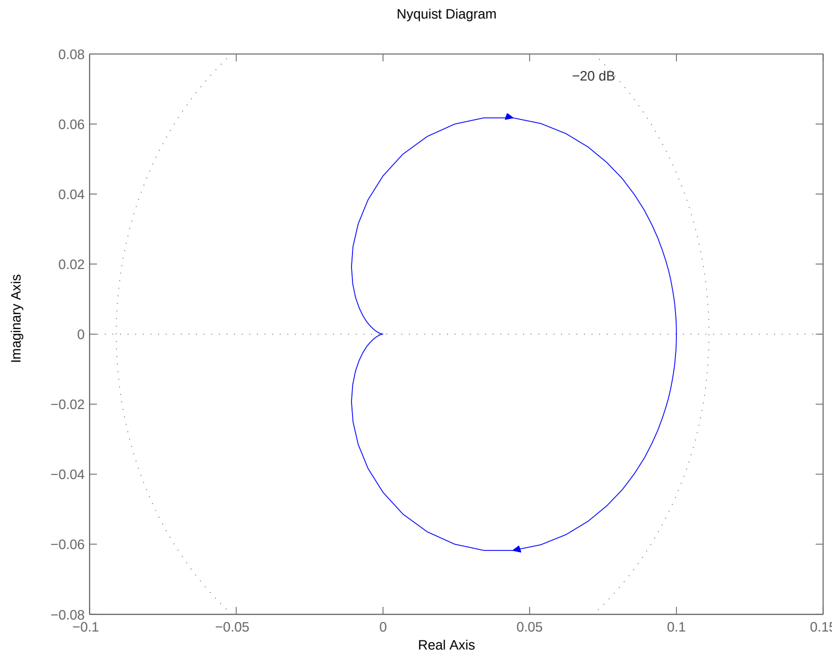
For the two plant transfer functions given below, use the Nyquist stability criterion to determine all values of the feedback gain K that stabilize the closed-loop system.

(a)

$$G(s) = \frac{1}{(s+2)(s+5)}$$



$$\omega \to 0 \Rightarrow |G(j\omega)| = \frac{1}{10}, \angle G(j\omega) = 0^\circ$$
$$\omega = \sqrt{10} \Rightarrow |G(j\omega)| = \frac{1}{7\sqrt{10}}, \angle G(j\omega) = -90^\circ$$
$$\omega \to \infty \Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -180^\circ$$

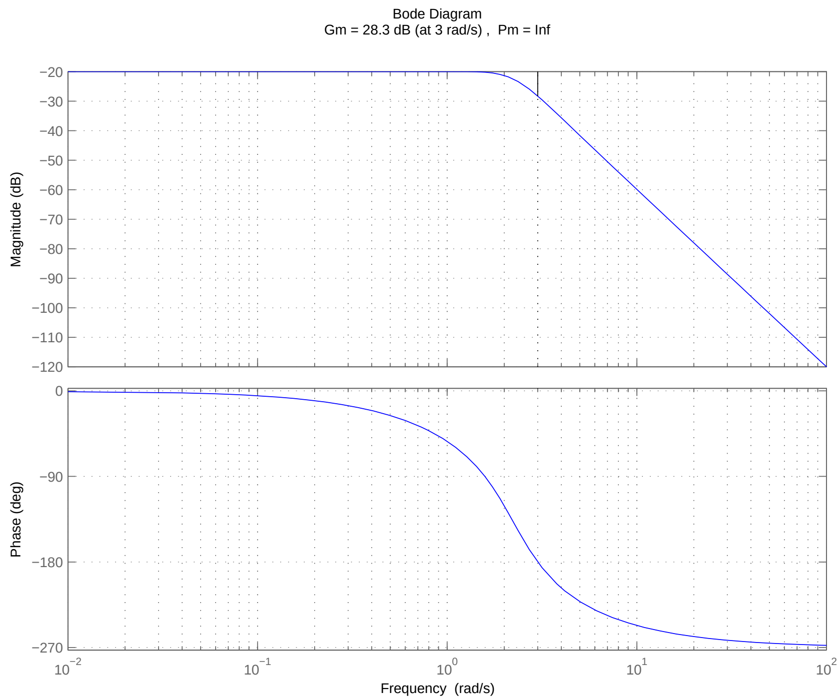


P: #RHP open loop poles = 0
Z: # of closed loop RHP poles
N: # of encirclements of $-1/K$
N = Z - P $\Rightarrow$ N = Z

If $K > 0 \Rightarrow -\dfrac{1}{K} < 0$, then $N = 0$ according to the Nyquist plot
If $-10 < K \leq 0 \Rightarrow -\dfrac{1}{K} > \dfrac{1}{10}$, then $N = 0$
If $K < -10 \leq 0 \Rightarrow 0 < -\dfrac{1}{K} < \dfrac{1}{10}, \Rightarrow N = 1 \Rightarrow Z = 1 \Rightarrow$ unstable closed-loop
Therefore, we need $K \geq -10$ for stability, which agrees with the Routh's criterion.

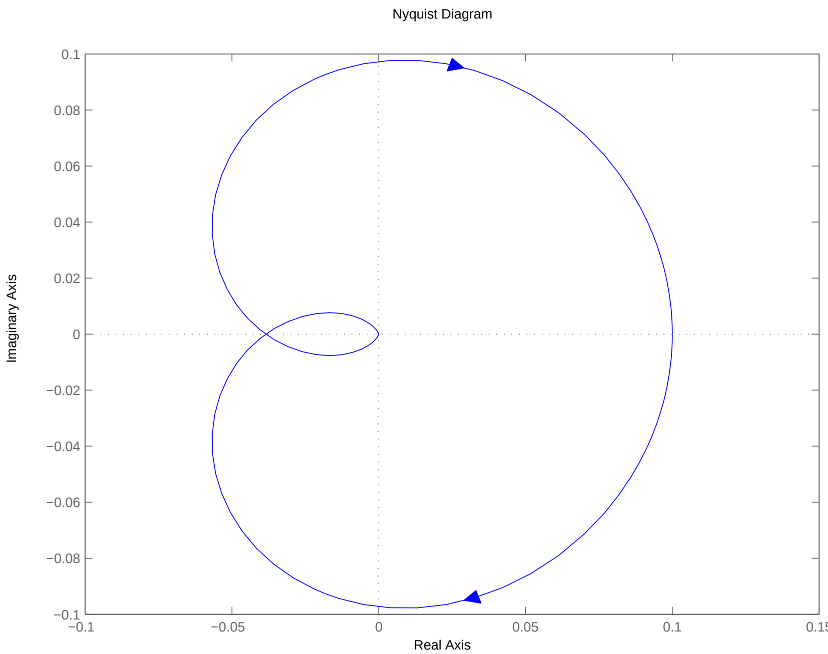
(b)

$$G(s) = \frac{1}{(s+2)(s^2+2s+5)}$$



From the Bode plot

$$
\begin{aligned}
\omega \to 0 &\Rightarrow |G(j\omega)| = \frac{1}{10}, \angle G(j\omega) = 0^\circ \\
\omega = \sqrt{\frac{5}{2}} &\Rightarrow |G(j\omega)| = 0.097, \angle G(j\omega) = -90^\circ \\
\omega = 3 &\Rightarrow |G(j\omega)| = 0.0385, \angle G(j\omega) = -180^\circ \\
\omega \to \infty &\Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -270^\circ
\end{aligned}
$$



$P = 0$
$\therefore Z = N$
Following a similar procedure as we did for part a, we will find that $-10 < K < 26$ yields $N = 0 \Rightarrow$ Stability

Check with Routh's Criterion
Characteristic Equation: $1 + KG(S) = 0 \Rightarrow s^3 + 4s^2 + 9s + 10 + K$
Necessary condition: $K > -10$

s^3	1	9
s^2	4	$10 + K$
s^1	$-\frac{1}{4}(10 + K - 36)$	
s^0	$10 + K$	

Therefore, $-\frac{1}{4}(10 + K - 36) > 0 \Rightarrow K < 26$

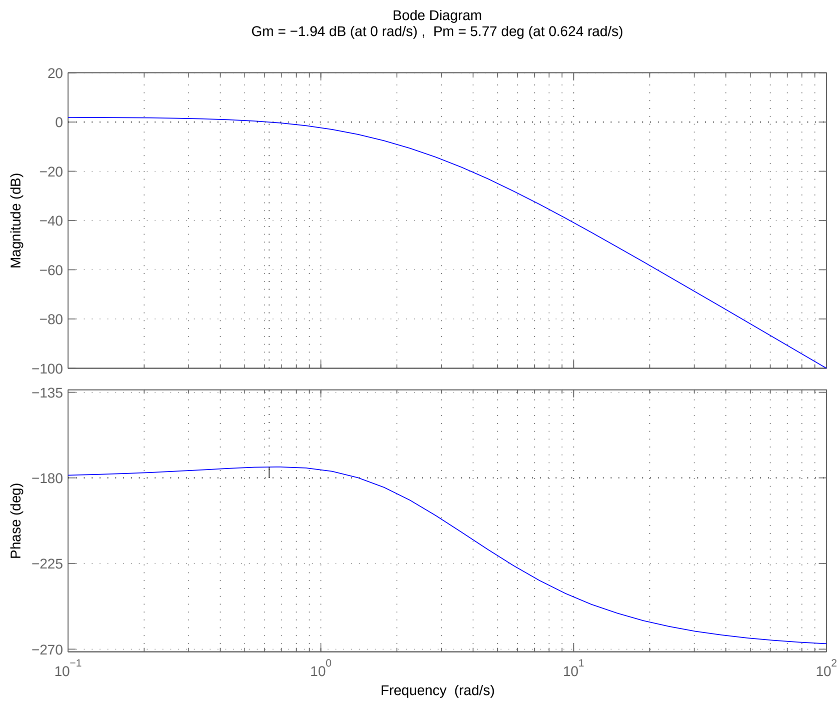
Problem 3

For the two transfer functions and gain values given below, use the Nyquist plot to find the gain and the phase margins:

(a) $G(s) = \dfrac{1}{(s-1)(s+2)(s+4)}, \quad K = 10$

$$KG_1(s) = \frac{10}{(s-1)(s+2)(s+4)} KG_1(j\omega) \qquad = \frac{10}{(j\omega-1)(j\omega+2)(j\omega+4)}$$

Bode plot of $KG_1(s)$



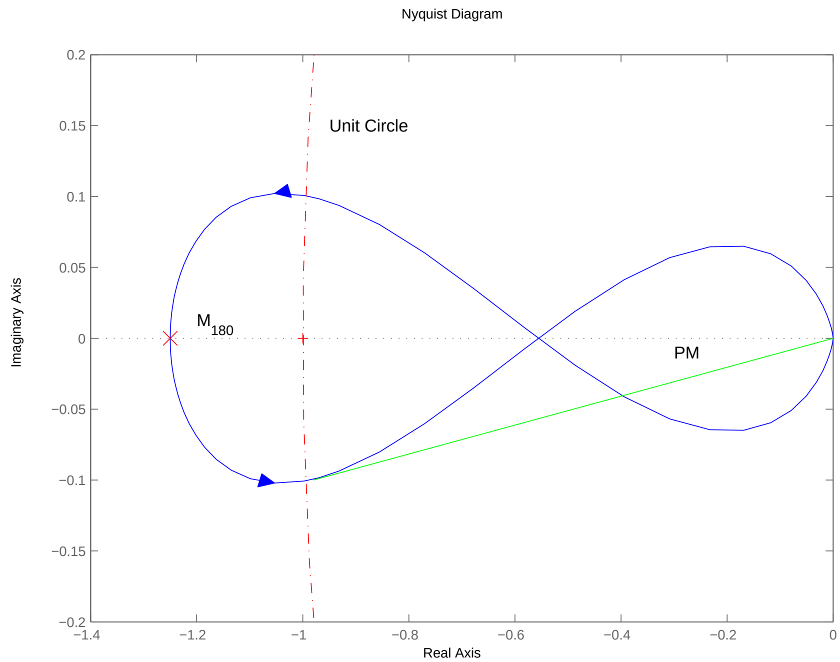
From the bode plot, we can see that

$\omega \to 0 \Rightarrow |G(j\omega)| = \dfrac{10}{8}, \angle G(j\omega) = -180°$

Therefore, $M_{180°} = \dfrac{10}{8} \Rightarrow GM = \dfrac{1}{M_{180°}} = \dfrac{8}{10} = 0.8 = -1.94$ dB, which agrees with the gain margin obtained from MATLAB.

To obtain PM from the Nyquist plot, draw a unit circle and marked the points where the unit circle intersects the Nyquist plot. Draw a line from the origin to one of the points. The angle formed between that line and the $-180°$ axis is the PM. See the plot below for an illustration of the method.

Nyquist plot of $KG_1(s)$



PM can be computed from the ω that makes $|KG_1(j\omega)| = 1$

$$\left|\frac{10}{(j\omega-1)(j\omega+2)(j\omega+4)}\right| = 1$$
$$\left|\frac{10}{(-5\omega^2-8)+(-\omega^3+2\omega)j}\right| = 1$$
$$\frac{10}{\sqrt{(-5\omega^2-8)^2+(-\omega^3+2\omega)^2}} = 1$$
$$(-5\omega^2-8)^2+(-\omega^3+2\omega)^2 = 100$$

Solve this equation and find that $\omega = 0.625$ satisfies this equation.
Substitute, $\omega = 0.625$ in the $\angle KG_1(j\omega)$, which will result in $\angle KG_1(j0.625) = -174.23°$.

$\therefore PM = -174.23° - (-180°) = 5.77°$, which agrees with the gain margin obtained from MATLAB.

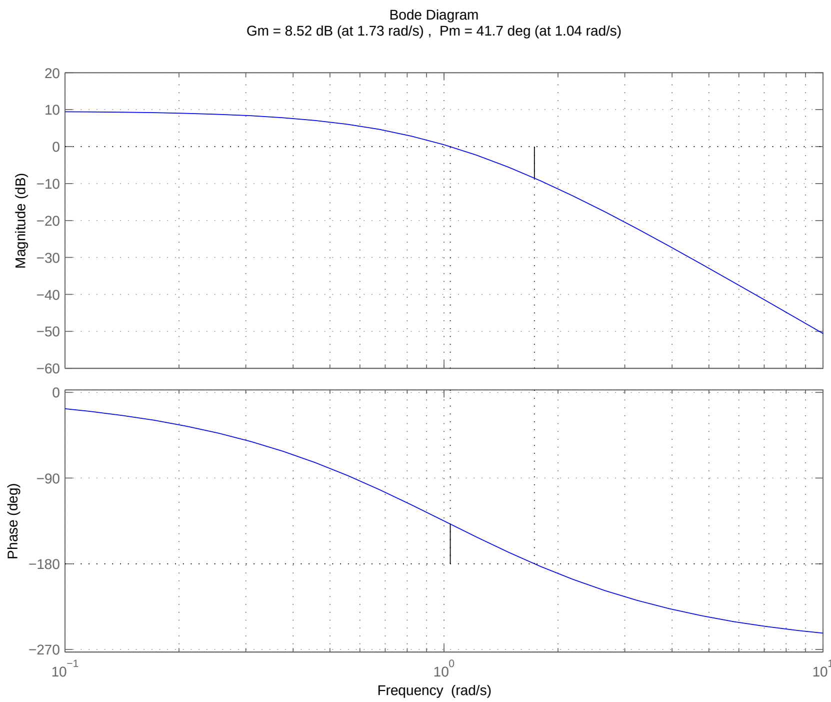
(b) $G(s) = \dfrac{1}{(s+1)^3}, \quad K = 3$

$$KG_2(s) = \frac{3}{(s+1)^3}$$

$$KG_2(j\omega) = \frac{3}{(j\omega+1)^3}$$

$$= \frac{3}{(1-3\omega^2)+(3\omega-\omega^3)j}$$

Bode plot of $KG_2(s)$



$$\omega \to 0 \Rightarrow |G(j\omega)| = 3, \angle G(j\omega) = 0^\circ$$

$$\omega = \frac{1}{\sqrt{3}} \Rightarrow |G(j\omega)| = 1.95, \angle G(j\omega) = -90^\circ$$

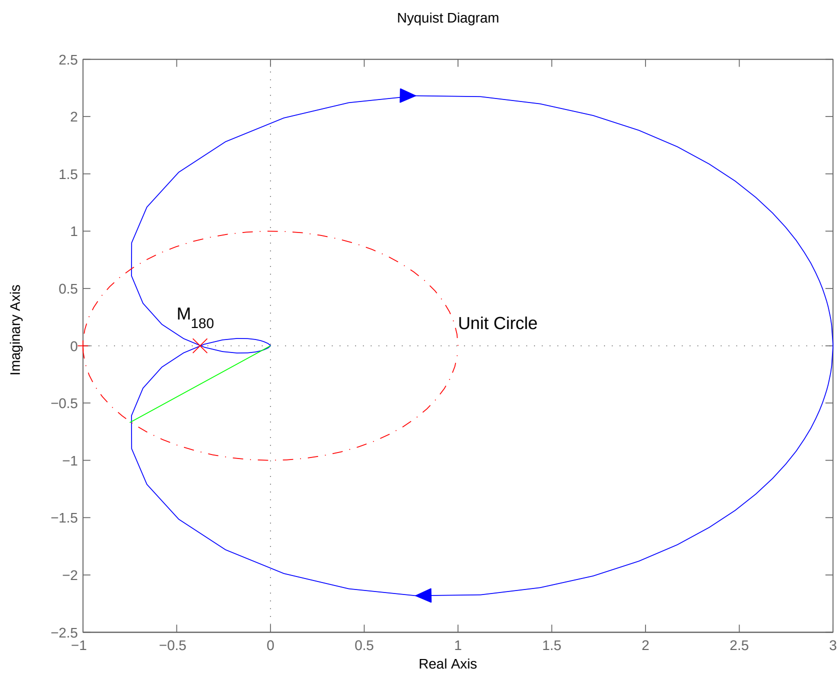
$$\omega = \sqrt{3} \Rightarrow |G(j\omega)| = 0.375, \angle G(j\omega) = -180^\circ$$

$$\omega \to \infty \Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -270^\circ$$

Therefore, $M_{180^\circ} = 0.375 \Rightarrow GM = \frac{1}{M_{180^\circ}} = \frac{1}{0.375} = 2.667 = 8.519$ dB, which agrees with the gain margin obtained from MATLAB.

Follow the same step as the previous problem, to obtain PM

Nyquist plot of $KG_2(s)$



Solve this equation and find that $\omega = 1.04$ satisfies this equation.
Substitute, $\omega = 1.04$ in the $\angle KG_2(j\omega)$, which will result in $\angle KG_2(j1.04) = -138.36°$.

$\therefore PM = -138.36° - (-180°) = 41.7°$, which agrees with the gain margin obtained from MATLAB.