Solution 1

Label the signals on the outlets of the first two summation junctions to be V, W, respectively. Establish equations around the three summation junctions:

$$V = U - K_2 P_2 P_1 W \tag{1}$$
$$W = K_1 V - K_2 P_2 W \tag{2}$$
$$Y = K_2 G W + K_2 P_2 P_1 W \tag{3}$$

Replace V in (2) by (1),

$$W = K_1 U - K_1 K_2 P_1 P_2 W - K_2 P_2 W \Rightarrow W = \frac{K_1}{1 + K_2 P_2 + K_1 K_2 P_1 P_2} U \tag{4}$$

Substitute (4) into (3), we have:

$$\frac{Y}{U} = \frac{K_1 K_2 G + K_1 K_2 P_1 P_2}{1 + K_2 P_2 + K_1 K_2 P_1 P_2}$$

Solution 2

(i) Through the forward plant block:

$$G = \frac{Y}{E} = c(s-a)^{-1} b = \frac{bc}{s-a}$$

Through the feedback block:

$$H = \frac{V}{Y} = 1(s-k)^{-1} l + m = \frac{l}{s-k} + m$$

Therefore,

$$\frac{Y}{R} = \frac{G}{1+GH} = \frac{\frac{bc}{s-a}}{1+\frac{bc}{s-a}\left(\frac{l}{s-k}+m\right)} = \frac{bc(s-k)}{s^2+(bcm-a-k)s+(ak+bcl-bckm)}$$

(ii) Notice that the denominator of the transfer function is a monic second order polynomial in the form

$$p(s) = s^2 + a_1 s + a_2 \tag{5}$$

In order for the system to be stable, we need all roots of (5) to be in the OLHP.

When the determinant of (5) is non-negative, that is, $a_1^2 - 4a_2 \geq 0$, we need both roots to be negative:

$$\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} < 0 \Rightarrow a_1 > 0, 0 < a_2 \leq \frac{a_1^2}{4};$$

When $a_1^2 - 4a_2 < 0$, we need the real part of the roots to be negative:

$$\frac{-a_1}{2} < 0 \Rightarrow a_1 > 0, a_2 > \frac{a_1^2}{4}$$

Hence it can be summarized that we need both $a_1 > 0, a_2 > 0$ in order for the system to be stable. This condition can be directly obtained by using Routh-Hurwitz criterion. Substitute a_1, a_2 with the system parameters given, all we need is:

$$\begin{cases} bcm - a - k > 0, \\ ak + bcl - bckm > 0 \end{cases}$$

Solution 3

(i) From the state space model we have:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{\rho}{m}x_2 + \frac{1}{m}u$$
$$y = x_1$$

Applying Laplace transform on all three equations above and omitting initial conditions:

$$sX_1 = X_2 \tag{6}$$
$$sX_2 = -\frac{k}{m}X_1 - \frac{\rho}{m}X_2 + \frac{1}{m}U \tag{7}$$
$$Y = X_1 \tag{8}$$

Substitute (8) into (6) we get

$$X_2 = sY \tag{9}$$

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Now replace X_1, X_2 in (7) by (8) and (9),

$$s^2 Y = -\frac{k}{m} Y - \frac{\rho}{m} s Y + \frac{1}{m} U$$

$$\Rightarrow \frac{Y}{U} = \frac{1}{ms^2 + \rho s + k}$$

which is the transfer function of the system.

Alternatively, we can directly derive the transfer function using the following formula:

$$G(s) = C(Is - A)^{-1} B$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\frac{1}{s^2 + \frac{\rho}{m} s + \frac{k}{m}} \begin{pmatrix} s + \frac{\rho}{m} & 1 \\ -\frac{k}{m} & s \end{pmatrix} \right) \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$$

$$= \frac{1}{ms^2 + \rho s + k}$$

(ii) Similar to (i), from the state space model we have:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m} x_1 - \frac{\rho}{m} x_2 + \frac{1}{m} u$$

$$y = c_1 x_1 + c_2 x_2$$

Applying Laplace transform on those equations and do the algebra we can derive the transfer function. Alternatively, directly from the state-space model we can conclude

$$\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \left(\frac{1}{s^2 + \frac{\rho}{m} s + \frac{k}{m}} \begin{pmatrix} s + \frac{\rho}{m} & 1 \\ -\frac{k}{m} & s \end{pmatrix} \right) \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$$

$$= \frac{\frac{c_2}{m} s + \frac{c_1}{m}}{s^2 + \frac{\rho}{m} s + \frac{k}{m}}$$

Therefore by comparison, we should have

$$\begin{array}{l} \frac{c_2}{m} = 0 \\ \frac{c_1}{m} = \frac{k}{m} \end{array} \Rightarrow \begin{array}{l} c_1 = k \\ c_2 = 0 \end{array}$$

In addition,

$$\begin{array}{l} 2\zeta\omega_n = \frac{\rho}{m} \\ \omega_n^2 = \frac{k}{m} \end{array} \Rightarrow \begin{array}{l} \omega_n = \sqrt{\frac{k}{m}} \\ \zeta = \frac{\rho}{2\sqrt{km}} \end{array}$$
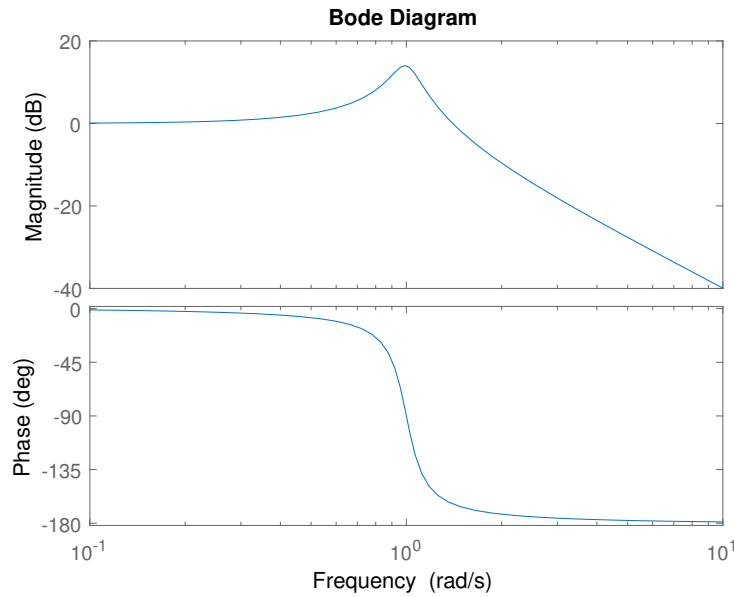
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(iii)

$$|G(j\omega)| = \frac{\frac{k}{m}}{|(j\omega)^2 + \frac{\rho}{m}j\omega + \frac{k}{m}|} = \frac{k}{\sqrt{(k - m\omega)^2 + (\rho\omega)^2}}$$

$$\angle G(j\omega) = -\angle((j\omega)^2 + \frac{\rho}{m}j\omega + \frac{k}{m}) = -\arctan\left(\frac{\rho\omega}{k - m\omega^2}\right)$$

Hence the frequency response will be:

$$y(t) = |G(j\omega)|\cos(\omega t + \angle G(j\omega)) = \frac{k}{\sqrt{(k - m\omega)^2 + (\rho\omega)^2}}\cos(\omega t - \arctan\left(\frac{\rho\omega}{k - m\omega^2}\right))$$

(iv) For illustration purpose, pick $\omega_n = 1$ and $\zeta = 0.1$. The plot can be either generated using the magnitude and phase formula derived in (iii) with a variable of ω, or by directly using Bode plot:

It can be seen that when $\omega \approx \omega_n$, magnitude increases drastically and reaches a maximum. Phase angle also drops rapidly near ω_n.
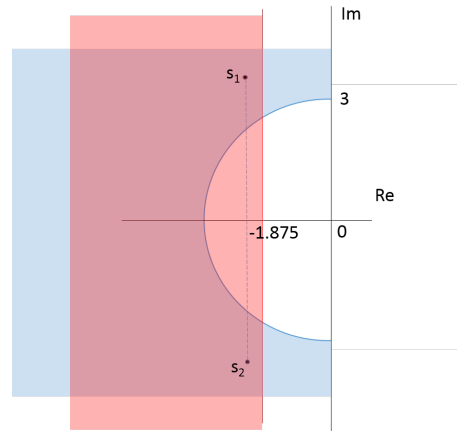
Solution 4

(i) Recall that

$$t_r = \frac{1.8}{\omega_n}, \quad t_s = \frac{3}{\sigma}$$

Therefore by the given specs $t_r \leq 0.6, ts \leq 1.6$, we can conclude that

$$\omega_n \geq 3, \sigma \geq 1.875$$

The admissible pole locations are shown in the figure below, where the blue shaded region is due to rise time constraint and red shaded region is due to the settling time constraint.

The poles of $H(s)$ are the roots of $s^2 + 4s + 16$, which can be computed to be $s_{1,2} = -2 \pm 2\sqrt{3}$. they are plotted in the complex plane and found to be in both of shaded regions. Hence the given system satisfy these specs.
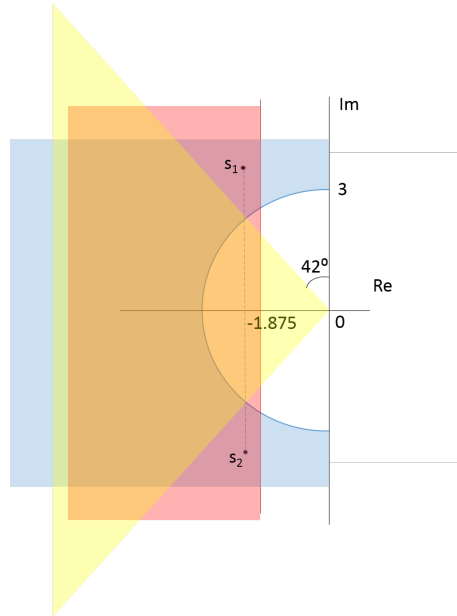
(ii) Recall that

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\pi \arctan\theta}$$

By the given spec of $M_p \leq 1/e^2$, it can be computed that

$$\theta \geq 0.74 \text{ rad} = 42°$$

The admissible region due to overshoot constraint is plotted as the yellow shaded area in the figure on the next page. Since the poles of the system are not in the yellow region, the given system does not satisfy the new spec.

(iii) Recall that

$$t_p = \frac{\pi}{\omega_d}$$

The given spec of $t_p \leq 1$ suggests $\omega_d \geq \pi$. Notice that in the complex plane the damped frequency ω_d represents the distance from a pole to the real-axis; therefore, this spec can be converted into the admissible region represented by shaded green area in the figure below:
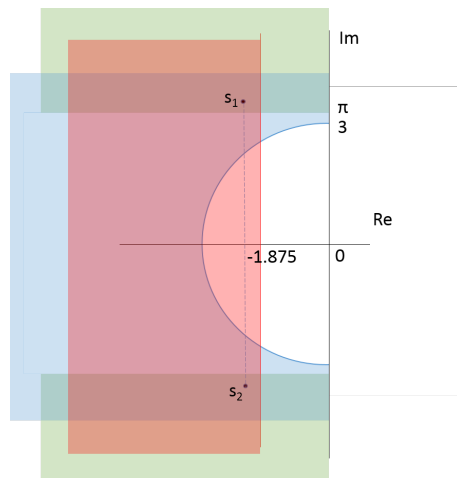
As the poles are in the green region, the given system satisfy the new spec.