

**Reading Assignment:**

FPE, Sections 6.4 (re-read), 7.1–7.4.

**Problems:**

1. Prove the following matrix identities (you may assume that all matrices have correct shapes, are invertible, etc.):

(i)  $(AB)^T = B^T A^T$

(ii)  $(ABC)^T = C^T B^T A^T$

(iii)  $(A^T)^{-1} = (A^{-1})^T$

(iv)  $(I - TAT^{-1})^{-1} = T(I - A)^{-1}T^{-1}$

(v) For any integer  $k \geq 0$ ,  $(TAT^{-1})^k = TA^kT^{-1}$

2. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \quad y = x_2.$$

- (i) Write down the open-loop characteristic equation. (This involves computing a  $3 \times 3$  determinant, which you can do either by hand or in MATLAB using a symbolic variable  $s$ .) Are all open-loop poles in the LHP?
- (ii) Using the formula given in class, compute the transfer function of this system. (Use the general formula, do *not* take Laplace transform of individual differential equations. Look up the procedure for inverting a matrix by hand, or use the MATLAB command `inv`.)
- (iii) Find a CCF realization of the same transfer function, in the form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = (b_1 \ b_2 \ b_3)x.$$

*Hint:* You should see that, similarly to the  $2 \times 2$  case discussed in class, there is a simple relation between the entries in the above matrices and the coefficients in the transfer function.

3. Determine (from the controllability matrix) whether or not the following systems are controllable:

(i)  $\dot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$       (ii)  $\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u.$

You can use MATLAB to perform matrix multiplication, but you should know how to do it by hand.

4. Imagine two cars driving on the same road in the same direction, one behind the other, and trying to go at the same speed. This situation can be described by the linear system

$$\begin{aligned}\dot{v}_1 &= -(v_1 - v_2) + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2\end{aligned}$$

where, for  $i = 1, 2$ ,  $v_i$  is the velocity of car  $i$  and  $f_i$  is an external force (wind, road conditions, etc.) acting on it. The meaning of the above equations is that each car accelerates/decelerates depending on whether it is going slower/faster than the other. Now, suppose that we want to rewrite the above system in the following new coordinates:  $\bar{v}_1 := v_1$  (velocity of car 1) and  $\bar{v}_2 := v_2 - v_1$  (relative velocity of the two cars).

- (i) Write down the differential equations for  $\dot{\bar{v}}_1, \dot{\bar{v}}_2$  in terms of  $\bar{v}_1, \bar{v}_2$  and  $f_1, f_2$ .
- (ii) Write down the original system in state-space form  $\dot{v} = Av + Bf$ , the new system in state-space form  $\dot{\bar{v}} = \bar{A}\bar{v} + \bar{B}f$ , the coordinate transformation matrix  $T$  from  $(v_1, v_2)$  to  $(\bar{v}_1, \bar{v}_2)$ , and verify that the formulas given in class (relating  $A$  with  $\bar{A}$  and  $B$  with  $\bar{B}$  via  $T$ ) hold.

*Hint:* since we have two inputs,  $B$  is a matrix, not a vector.