Plan of the Lecture

- Review: transient response specs (rise time, overshoot, settling time)
- Today's topic: effect of zeros and extra poles; Routh–Hurwitz stability criterion

Goal: understand the effect of zeros and high-order poles on the shape of transient response; discuss relation with stability; formulate and learn how to apply the Routh–Hurwtiz stability criterion.

Reading: FPE, Sections 3.5–3.6

Effect of Zeros on the Transient Response

Reminder: for $H(s) = \frac{q(s)}{p(s)}$, zeros are the roots of $q(s) = 0$

Example: start with $H_1(s) = \frac{1}{s^2 + 2\zeta s + 1}$ $(\omega_n = 1)$

Let's add a zero at $s = -a$, $a > 0$ – LHP zero

To keep DC gain $= 1$, let's take the numerator to be $\frac{s}{a} + 1$:

$$
\begin{aligned}
H_2(s) &= \frac{\frac{s}{a} + 1}{s^2 + 2\zeta s + 1} \\
&= \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{\text{this is } H_1(s)} + \frac{1}{a} \cdot \underbrace{\frac{s}{s^2 + 2\zeta s + 1}}_{\text{call this } H_d(s)} \\
&= H_1(s) + \frac{1}{a} H_d(s), \qquad H_d(s) = s H_1(s)
\end{aligned}
$$

Effect of a LHP Zero

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s=-a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

Step response:

$$\begin{aligned} Y_1(s) &= \frac{H_1(s)}{s} \\ Y_2(s) &= \frac{H_2(s)}{s} \\ &= \frac{H_1(s)}{s} + \frac{1}{a}\frac{sH_1(s)}{s} \\ &= Y_1(s) + \frac{1}{a} sY_1(s) \end{aligned}$$

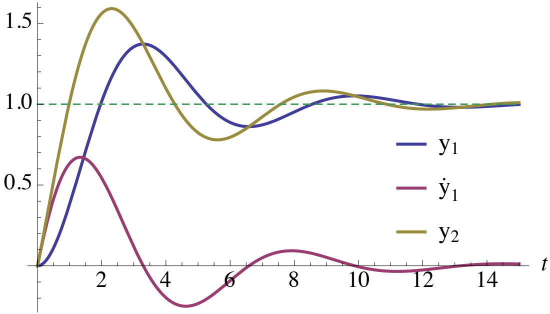
$$y_2(t) = \mathscr{L}^{-1}\{Y_2(s)\} = \mathscr{L}^{-1}\left\{Y_1(s) + \frac{1}{a} \cdot sY_1(s)\right\} = y_1(t) + \frac{1}{a}\dot{y}_1(t)$$

(assuming zero initial conditions)

Effect of a LHP Zero

Step response (zero at $s = -a$)

$$y_2(t) = y_1(t) + \frac{1}{a}\dot{y}_1(t) \qquad \text{where } y_1(t) = \text{original step response}$$



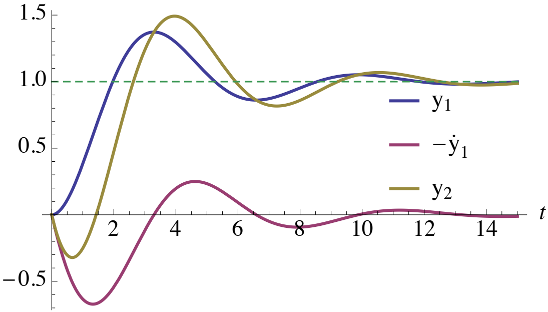
Effects of a LHP zero:

- increased overshoot (major effect)
- little influence on settling time
- what happens as $a \to \infty$? — effects become less significant

What About a RHP Zero?

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = a} H_2(s) = H_1(s) - \frac{1}{a} \cdot s H_1(s)$$

$$y_2(t) = y_1(t) - \frac{1}{a} \cdot \dot{y}_1(t)$$

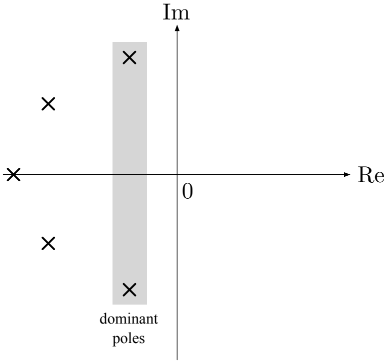


Effects of a RHP zero:

- slows down (delays) the response
- creates *undershoot* (at least, when a is small enough)

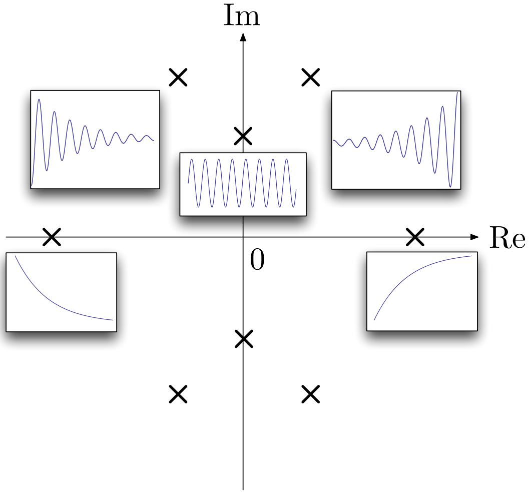
Effect of Extra Poles

A general nth-order system has n poles



- extra LHP poles are not significant if their real parts are at least 5× the real parts of dominant LHP poles
- e.g., if dominant poles have $\mathrm{Re}(s) = -2$ and we have extra poles with $\mathrm{Re}(s) = -10$, their time-domain contributions will be e^{-2t} and $e^{-10t} \ll e^{-2t}$
- 5× is just a convention, but we can really see the effect of extra poles that are closer (cf. Lab 2)

Effect of Pole Locations



- poles in open LHP ($\text{Re}(s) < 0$) — stable response
- poles in open RHP ($\text{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\text{Re}(s) = 0$) — tricky case

Marginal Case: Poles on the Imaginary Axis

Let's consider the case of a pole at the origin: $H(s) = \dfrac{1}{s}$

Is this a stable system?

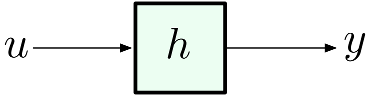
- impulse response: $Y(s) = \dfrac{1}{s} \Longrightarrow y(t) = 1(t) \quad \text{(OK)}$
- step respone: $Y(s) = \dfrac{1}{s^2} \Longrightarrow y(t) = t,\ t \geq 0$ — *unit ramp*!!

What about purely imaginary poles? $\quad H(s) = \dfrac{\omega^2}{s^2 + \omega^2}$

- impulse response: $Y(s) = \dfrac{\omega^2}{s^2 + \omega^2} \Longrightarrow y(t) = \omega \sin(\omega t)$
- step respone: $Y(s) = \dfrac{\omega^2}{s(s^2 + \omega^2)} \Longrightarrow y(t) = 1 - \cos(\omega t)$

Systems with poles on the imaginary axis are *not stable*.

What Is Stability?



One reasonable definition is as follows:

A linear time-invariant system is *Bounded-Input, Bounded-Output* (BIBO) *stable* provided either one of the following three equivalent conditions is satisfied:

1. If every bounded input $u(t)$ results in a bounded output $y(t)$, regardless of initial conditions.
2. If the impulse response $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty.$$

3. If all poles of the transfer function $H(s)$ are *strictly stable* (lie in open LHP).

Checking for Stability?

Consider a general transfer function:

$$H(s) = \frac{q(s)}{p(s)}$$

where q and p are polynomials, and $\deg(q) \leq \deg(p)$.

We need tools for checking stability: whether or not all roots of $p(s) = 0$ lie in OLHP.

For simple polynomials, can just factor them "by inspection" and find roots.

Now, this is hard to do for high-degree polynomials — it's computationally intensive, especially symbolically.

But: often we *don't need to know* precise pole locations, just need to know that they are strictly stable.

Checking for Stability

Problem: given an nth-degree polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

with real coefficients, check that the roots of the equation $p(s) = 0$ are strictly stable (i.e., have negative real parts).

Terminology: we often say that the polynomial p is (strictly) stable if all of its roots are.

A Necessary Condition for Stability

Terminology: we say that A is a necessary condition for B if

$$A \text{ is false} \quad \Longrightarrow \quad B \text{ is false}$$

Important!! Even if A is true, B may still be false.

Necessary condition for stability: a polynomial p is strictly stable only if all of its coefficients are strictly positive.

Proof: suppose that p has roots at $r_1, r_2, \ldots, r_n$ with $\mathrm{Re}(r_i) < 0$ for all i. Then

$$p(s) = (s - r_1)(s - r_2) \ldots (s - r_n)$$

— multiply this out and check that all coefficients are positive.

Routh–Hurwitz Criterion

Necessary & Sufficient Condition for Stability

Terminology: we say that A is a sufficient condition for B if

$$A \text{ is true} \quad \Longrightarrow \quad B \text{ is true}$$

Thus, A is a necessary *and* sufficient condition for B if

$$A \text{ is true} \quad \Longleftrightarrow \quad B \text{ is true}$$

— we also say that A is true if and only if (iff) B is true.

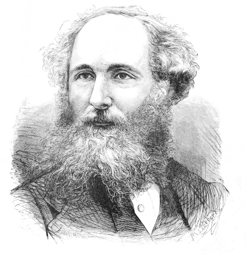
We will now introduce a necessary and sufficient condition for stability: the *Routh–Hurwitz Criterion*.

Routh–Hurwitz Criterion: A Bit of History

J.C. Maxwell, "On governors," Proc. Royal Society, no. 100, 1868

... [Stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative. ...

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.



In 1877, Maxwell was one of the judges for the Adams Prize, a biennial competition for best essay on a scientific topic. The topic that year was stability of motion. The prize went to Edward John Routh, who solved the problem posed by Maxwell in 1868.

In 1893, Adolf Hurwitz solved the same problem, using a different method, independently of Routh.

Edward John Routh, 1831–1907

Adolf Hurwitz, 1859–1919

Routh's Test

Problem: check whether the polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

is strictly stable.

We begin by forming the Routh array using the coefficients of p:

$$\begin{array}{rccccc} s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \end{array}$$

(if necessary, add zeros in the second row to match lengths)

Note that the very first entry is always 1, and also note the order in which the coefficients are filled in.

Routh's Test

$$\begin{array}{rllll} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \end{array}$$

Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

where

$$b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1} (a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1} (a_5 - a_1 a_4)$$

$$b_3 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1} (a_7 - a_1 a_6) \qquad \text{and so on ...}$$

Note: the new row is 1 element shorter than the one above it

Routh's Test, continued

$$\begin{array}{rccccc} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots & \\ s^{n-3}: & c_1 & c_2 & \dots & & \end{array}$$

Next, we form the fourth row marked by s^{n-3}:

$$s^{n-3}: \quad c_1 \quad c_2 \quad \dots$$

$$\text{where} \quad c_1 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_2 - a_3 b_1)$$

$$c_2 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_3 - a_5 b_1)$$

and so on ...

Routh's Test, continued

Eventually, we complete the array like this:

$$\begin{array}{rlllll} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots & \\ s^{n-3}: & c_1 & c_2 & \dots & & \\ \vdots & & & & & \\ s^1: & * & * & & & \\ s^0: & * & & & & \end{array}$$

(as long as we don't get stuck with division by zero: more on this later)

After the process terminates, we will have $n+1$ entries in the first column.

The Routh–Hurwitz Criterion

Consider degree-n polynomial

$$p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

and form the Routh array:

$$\begin{array}{rccccc}
s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2}: & b_1 & b_2 & b_3 & \ldots & \\
s^{n-3}: & c_1 & c_2 & \ldots & & \\
\vdots & & & & & \\
s^1: & * & * & & & \\
s^0: & * & & & &
\end{array}$$

The Routh–Hurwitz criterion: Assume that the necessary condition for stability holds, i.e., $a_1, \ldots, a_n > 0$. Then:

- p is stable if and only if all entries in the first column are positive;
- otherwise, #(RHP poles) = #(sign changes in 1st column)

Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

All coefficients strictly positive: necessary condition checks out.

$$\begin{array}{lccc} s^4: & 1 & 1 & 3 \\ s^3: & 4 & 2 & 0 \\ s^2: & 1/2 & 3 & \\ s^1: & -22 & 0 & \\ s^0: & 3 & & \end{array}$$

Answer: p is unstable — it has 2 RHP poles (2 sign changes in 1st column)

Low-Order Cases ($n = 2, 3$)

$n = 2$ $\qquad p(s) = s^2 + a_1 s + a_2$

$$\begin{array}{lll} s^2 & : 1 & a_2 \\ s^1 & : a_1 & 0 \\ s^0 : & b_1 & \end{array} \qquad b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & 0 \end{pmatrix} = a_2$$

— p is stable iff $a_1, a_2 > 0$ (necessary *and* sufficient).

$n = 3$ $\qquad p(s) = s^3 + a_1 s^2 + a_2 s + a_3$

$$\begin{array}{lll} s^3 & : 1 & a_2 \\ s^2 & : a_1 & a_3 \\ s^1 : & b_1 & 0 \\ s^0 : & c_1 & \end{array} \qquad \begin{array}{l} b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = \frac{a_1 a_2 - a_3}{a_1} \\ c_1 = -\frac{1}{b_2} \det \begin{pmatrix} a_1 & a_3 \\ b_1 & 0 \end{pmatrix} = a_3 \end{array}$$

— p is stable iff $a_1, a_2, a_3 > 0$ (necc. cond.) and $a_1 a_2 > a_3$

Stability Conditions for Low-Order Polynomials

The upshot:

- A 2nd-degree polynomial $p(s) = s^2 + a_1 s + a_2$ is stable if and only if $a_1 > 0$ and $a_2 > 0$
- A 3rd-degree polynomial $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$ is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

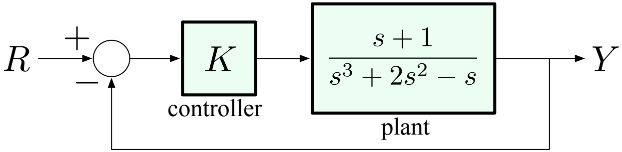
- These conditions were already obtained by Maxwell in 1868.
- In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

Routh–Hurwitz as a Design Tool

Parametric stability range

We can use the Routh test to determine *parameter ranges* for stability.

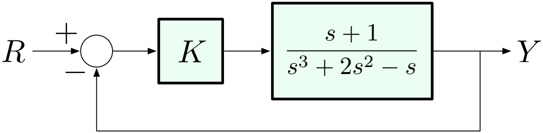
Example: consider the unity feedback configuration



Note that the plant is *unstable* (the denominator has a negative coefficient and a zero coefficient).

Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Example, continued



Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Let's write down the transfer function from R to Y:

$$\begin{aligned}\frac{Y}{R} &= \frac{\text{forward gain}}{1 + \text{loop gain}} \\ &= \frac{K \cdot \frac{s+1}{s^3+2s^2-s}}{1 + K \cdot \frac{s+1}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3+2s^2-s+K(s+1)} \\ &= \frac{Ks+K}{s^3+2s^2+(K-1)s+K}\end{aligned}$$

Now we need to test stability of $p(s) = s^3 + 2s^2 + (K-1)s + K$.

Example, continued

Test stability of

$$p(s) = s^3 + 2s^2 + (K-1)s + K$$

using the Routh test.

Form the Routh array:

$$\begin{array}{lcc} s^3: & 1 & K-1 \\ s^2: & 2 & K \\ s^1: & \frac{K}{2}-1 & 0 \\ s^0: & K & \end{array}$$

For p to be stable, all entries in the 1st column must be positive:

$$K > 2 \quad \text{and} \quad K > 0 \quad \text{(already covered by } K > 1\text{)}$$

Note: The necessary condition requires $K > 1$, but now we actually know that we must have $K > 2$ for stability.

Some Comments on the Routh Test

- The result (#(RHP roots)) is not affected if we multiply or divide any row of the Routh array by an arbitrary *positive* number.
- If we get a zero element in the 1st column, we can't continue. In that case, we can replace the 0 by a small number ε and apply Routh test to that. When we are done with the array, take the limit as $\varepsilon \to 0$. (see Ex. 3.33 in FPE)
- For an *entire row of zeros*, the procedure is a more complicated (see Example 3.34 in FPE) – we will not worry about this too much.