Plan of the Lecture

- **Review:** Bode plots for three types of transfer functions
- **Today’s topic:** stability from frequency response; gain and phase margins

**Goal:** learn to read off stability properties of the closed-loop system from the Bode plot of the open-loop transfer function; define and calculate Gain and Phase Margins, important quantitative measures of “distance to instability.”

*Reading:* FPE, Section 6.1
Consider this unity feedback configuration:

![Diagram of a unity feedback configuration]

**Question:** How can we decide whether the closed-loop system is stable for a given value of $K > 0$ based on our knowledge of the open-loop transfer function $KG(s)$?
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One answer: use root locus.

Points on the root locus satisfy the characteristic equation

$$1 + KG(s) = 0 \iff KG(s) = -1 \iff G(s) = -\frac{1}{K}$$

If $s \in \mathbb{C}$ is on the RL, then

$$|KG(s)| = 1 \text{ and } \angle KG(s) = \angle G(s) = 180^\circ \pmod{360^\circ}$$
Stability from Frequency Response

Question: How can we decide whether the closed-loop system is stable for a given value of $K > 0$ based on our knowledge of the open-loop transfer function $KG(s)$?

Another answer: let's look at the Bode plots:

$\omega \mapsto |KG(j\omega)|$ on log-log scale

$\omega \mapsto \angle KG(j\omega)$ on log-linear scale

— Bode plots show us magnitude and phase, but only for $s = j\omega$, $0 < \omega < \infty$

How does this relate to the root locus? $j\omega$-crossings!!
Stability from frequency response. If \( s = j\omega \) is on the root locus (for some value of \( K \)), then

\[
|KG(j\omega)| = 1 \quad \text{and} \quad \angle KG(j\omega) = 180^\circ \mod 360^\circ
\]

Therefore, the transition from stability to instability can be detected in two different ways:

- from root locus — as \( j\omega \)-crossings
- from Bode plots — as \( M = 1 \) and \( \phi = 180^\circ \) at some frequency \( \omega \) (for a given value of \( K \))
Example

\[ KG(s) = \frac{K}{s(s^2 + 2s + 2)} \]

Characteristic equation:

\[
1 + \frac{K}{s(s^2 + 2s + 2)} = 0 \\
\]

\[
s(s^2 + 2s + 2) + K = 0 \\
s^3 + 2s^2 + 2s + K = 0 
\]

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial \( s^3 + a_1 s^2 + a_2 s + a_3 \):

\[
a_1, a_2, a_3 > 0, \quad a_1 a_2 > a_3. \]

Here, the closed-loop system is stable if and only if \( 0 < K < 4 \). Let’s see what we can read off from the Bode plots.
Example, continued

$$KG(s) = \frac{K}{s(s^2 + 2s + 2)}$$

Bode form: $$KG(j\omega) = \frac{K}{2j\omega \left( (\frac{j\omega}{\sqrt{2}})^2 + j\omega + 1 \right)}$$

Plot the magnitude first:

- **Type 1 (low-frequency) asymptote:** $\frac{K/2}{j\omega}$
  
  $K_0 = K/2$, $n = -1 \implies$ slope $= -1$, passes through $(\omega = 1, M = K/2)$

- **Type 3 (complex pole) asymptote:**
  break-point at $\omega = \sqrt{2} \implies$ slope down by 2

- $\zeta = \frac{1}{\sqrt{2}} \implies$ no resonant peak
Example, Magnitude Plot

\[ KG(j\omega) = \frac{K}{2j\omega \left( \left( \frac{j\omega}{\sqrt{2}} \right)^2 + j\omega + 1 \right)} \]

Magnitude plot for \( K = 4 \) (the critical value):

When \( \omega = \sqrt{2} \), \( M = |4G(j\omega)| = \left| \frac{2}{j\sqrt{2}(j^2 + j\sqrt{2} + 1)} \right| = 1 \)
Example, Phase Plot

\[ KG(j\omega) = \frac{K}{2j\omega \left( \left( \frac{j\omega}{\sqrt{2}} \right)^2 + j\omega + 1 \right)} \]

Phase plot (independent of \( K \)):

When \( \omega = \sqrt{2} \), \( \phi = -180^\circ \)
For the critical value $K = 4$:

$M = 1$ and $\phi = 180^\circ \mod 360^\circ$ at $\omega = \sqrt{2}$
Crossover Frequency and Stability

**Definition:** The frequency at which \( M = 1 \) is called the *crossover frequency* and denoted by \( \omega_c \).

Transition from *stability* to *instability* on the Bode plot:

for critical \( K \), \( \angle G(j\omega_c) = 180^\circ \)
Effect of Varying $K$

What happens as we vary $K$?

- $\phi$ independent of $K \implies$ only the $M$-plot changes

- If we multiply $K$ by 2:
  \[
  \log(2M) = \log 2 + \log M
  \]
  - $M$-plot shifts up by $\log 2$

- If we divide $K$ by 2:
  \[
  \log\left(\frac{1}{2}M\right) = \log \frac{1}{2} + \log M
  \]
  \[
  = - \log 2 + \log M
  \]
  - $M$-plot shifts down by $\log 2$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!
Effect of Varying $K$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!

What happens as we vary $K$?

$$\angle KG(j\omega_c) \begin{cases} > -180^\circ, & \text{for } K < 4 \\ = -180^\circ, & \text{for } K = 4 \\ < -180^\circ, & \text{for } K > 4 \end{cases}$$

(stable) (critical) (unstable)
Effect of Varying $K$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!

Equivalently, we may define $\omega_{180^\circ}$ as the frequency at which

$$\phi = 180^\circ \mod 360^\circ.$$  

Then, in this example*,

$$|KG(j\omega_{180^\circ})| < 1 \iff \text{stability}$$

$$|KG(j\omega_{180^\circ})| > 1 \iff \text{instability}$$

* Not a general rule; conditions will vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.
Consider this unity feedback configuration:

\[
R \xrightarrow{+} K \xrightarrow{G(s)} Y
\]

Suppose that the closed-loop system, with transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

is stable for a given value of \( K \).

**Question:** Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).
Gain Margin

Back to our example: \[ G(s) = \frac{1}{s(s^2 + 2s + 2)}, \quad K = 2 \text{ (stable)} \]

Gain margin (GM) is the factor by which \( K \) can be multiplied before we get \( M = 1 \) when \( \phi = 180^\circ \)

Since varying \( K \) doesn’t change \( \omega_{180^\circ} \), to find GM we need to inspect \( M \) at \( \omega = \omega_{180^\circ} \)
Gain Margin

Our example: \( G(s) = \frac{1}{s(s^2 + 2s + 2)}, \ K = 2\) (stable)

Gain margin (GM) is the factor by which \( K \) can be multiplied before we get \( M = 1 \) when \( \phi = 180^\circ \). Since varying \( K \) doesn’t change \( \omega_{180^\circ} \), to find GM we need to inspect \( M \) at \( \omega = \omega_{180^\circ} \).

In this example:

\[
\omega_{180^\circ} = \sqrt{2}
\]

\( M = 0.5 (-6 \text{ dB}) \), so GM = 2
Phase Margin

Our example: \( G(s) = \frac{1}{s(s^2 + 2s + 2)} \), \( K = 2 \) (stable)

Phase margin (PM) is the amount by which the phase at the crossover frequency \( \omega_c \) differs from \( 180^\circ \) mod \( 360^\circ \)

To find PM, we need to inspect \( \phi \) at \( \omega = \omega_c \)

In this example:

\[ \phi \approx -148^\circ \]

\( M = 1 \)

\( \omega_c \approx 0.92 \)

at \( \omega_c \approx 0.92 \)

\[ \phi = -148^\circ, \]

so \( PM = (-148^\circ) - (-180^\circ) = 32^\circ \)

(in practice, want \( PM \geq 30^\circ \))
Consider gain \( K = 1 \), which gives closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

— prototype 2nd-order response

**Question:** what is the gain margin at \( K = 1 \)?

**Answer:** \( GM = \infty \)
Example 2

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

Let’s look at the phase plot:

- starts at \(-90^\circ\) (Type 1 term with \(n = -1\))
- goes down by \(90^\circ\) (Type 2 pole)

Recall: to find GM, we first need to find \(\omega_{180^\circ}\), and here there is no such \(\omega \implies \) no GM.
So, at $K = 1$, the gain margin of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

is equal to $\infty$ — what does that mean?

It means that we can keep on increasing $K$ indefinitely without ever encountering instability.

What about phase margin?
Example 2: Phase Margin

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left( \frac{j\omega}{2\zeta\omega_n} + 1 \right)} \]

Let’s look at the magnitude plot:

- low-frequency asymptote slope $-1$ (Type 1 term, $n = -1$)
- slope down by 1 past the breakpt. $\omega = 2\zeta\omega_n$ (Type 2 pole)

$\implies$ there is a finite crossover frequency $\omega_c$!!
Example 2: Magnitude Plot

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left( \frac{j\omega}{2\zeta\omega_n} + 1 \right)} \]

It can be shown that, for this system,

\[ \text{PM}\bigg|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right) \]

— for \( \text{PM} < 70^\circ \), a good approximation is \( \text{PM} \approx 100 \cdot \zeta \)
Phase Margin for 2nd-Order System

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

\[ \left. \text{PM} \right|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}}\right) \approx 100 \cdot \zeta \]

Conclusions:

larger PM \iff better damping
(open-loop quantity) \iff (closed-loop characteristic)

Thus, the overshoot \( M_p = \exp \left( -\frac{\pi\zeta}{\sqrt{1-\zeta^2}} \right) \) and resonant peak \( M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \) are both related to PM through \( \zeta \)!!
In the next lecture, we will see the following more generally:

**Bode’s Gain-Phase Relationship**: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a **design guideline**.