ECE 486: Control Systems

Lecture 15A: Bode Plots
A Bode plot for an LTI system $G(s)$ consists of two subplots:

- Magnitude (Gain) vs. frequency and
- Phase vs. frequency.

Such plots are useful to understand the steady-state response of the system $G(s)$ to sinusoids of different frequencies.
Bode Plots

If a stable, LTI system $G(s)$ is forced by $u(t) = \sin(\omega t)$ then:

$$y(t) \rightarrow |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

A Bode Plot is a common tool used to understand a linear system to sinusoids at different frequencies. It consists of two subplots:

- **Bode Magnitude (Gain) Plot**: Gain vs. Frequency
  - Horizontal axis is $\omega$ on a log (base 10) scale in units of rad/sec.
  - Vertical axis is the gain in decibels: $\frac{G(j\omega)}{20} = 20 \log_{10} |G(j\omega)|$
  - We can convert from dB back to actual gain: $G(j\omega) = 10^{\frac{G(j\omega)}{20}}$

<table>
<thead>
<tr>
<th>$G(j\omega)$</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
<th>$\frac{1}{\sqrt{2}}$</th>
<th>1</th>
<th>$\sqrt{2}$</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>G(j\omega)</td>
<td>_{dB}$</td>
<td>-60</td>
<td>-40</td>
<td>-20</td>
<td>-12</td>
<td>-6</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>20</td>
</tr>
</tbody>
</table>

- **Bode Phase Plot**: Phase vs. Frequency
  - Horizontal axis is $\omega$ on a log (base 10) scale in units of rad/sec.
  - Vertical axis is the phase $\angle G(j\omega)$ in degrees.
Example

System: \( G(s) = \frac{2}{s+4} \)

- If \( \omega \approx 0 \) then \( |G(j\omega)| \approx 0.5 \) and \( \angle G(j\omega) \approx 0^\circ \) so:
  
  If \( u(t) = \sin(\omega t) \) then \( y(t) \to 0.5 \sin(\omega t) \)

- If \( \omega \to \infty \) then \( |G(j\omega)| \to 0 \) and \( \angle G(j\omega) \approx -90^\circ \)

- If \( \omega = \frac{2 \text{ rad}}{\text{sec}} \) then \( |G(j\omega)| = 0.45 \) and \( \angle G(j\omega) = -27^\circ \)

\[ \text{MATLAB: } \quad \text{G = tf([2, 0], [1, 4]);} \]
\[ \text{G = bode(G);} \]
Bode Plot: Differentiator

- **Differentiator:** \( y(t) = \dot{u}(t) \)
  - \( G(s) = \frac{s}{1} = s \)
  - If \( u(t) = \sin(\omega t) \) then \( y(t) = \omega \cos(\omega t) = \omega \sin(\omega t + \frac{\pi}{2}) \)
  - This agrees with \(|G(j \omega)| = \omega\) and \( \angle G(j \omega) = \frac{\pi}{2}\text{rad} = 90^\circ \)

- **Properties:**
  - Differentiator amplifies higher frequencies and output leads input.
  - Slope of magnitude plot is +20dB/decade
Bode Plot: Integrator

- Integrator: \( \dot{y}(t) = u(t) \)  \( G(s) = \frac{1}{s} \)
- If \( u(t) = \sin(\omega t) \) then \( y(t) = -\frac{1}{\omega} \cos(\omega t) \)

[Neglecting a constant term and effect of initial conditions]
Bode Plot: Integrator

- Integrator: \[ \dot{y}(t) = u(t) \]
  \[ G(s) = \frac{1}{s} \]
  - If \( u(t) = \sin(\omega t) \) then \( y(t) = \frac{1}{\omega} \sin(\omega t - \frac{\pi}{2}) \)
  - This agrees with \(|G(j\omega)| = \frac{1}{\omega}\) and \(\angle G(j\omega) = -\frac{\pi}{2}\) rad = \(-90^\circ\)

- Properties:
  - Integrator amplifies lower frequencies and output lags input.
  - Slope of magnitude plot is -20dB/decade.
Now we focus on Bode plots for first order systems.

The Bode plot for \( G(s) = \frac{b_0}{s+a_0} \) has the following key features:

- The pole defines a corner frequency \( (\omega = |a_0|) \) for the system.
- The magnitude is flat at low frequencies and rolls off at −20dB per decade at high frequencies.
- The phase transitions by \( \pm 90^\circ \) near the corner frequency with precise details depending on the signs of \( (b_0, a_0) \).

The Bode plot for \( G(s) = \frac{s+b_0}{a_0} \) has the similar features except:

- The zero defines a corner frequency \( (\omega = |b_0|) \) for the system.
- The magnitude rolls up at +20dB per decade at high frequencies.
Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad G(s) = \frac{b_0}{s + a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

Bode plots can be generated by Matlab:

```matlab
>> G = tf([2], [1 4]);
>> bode(G);
```

It will be useful to sketch straight-line approximate Bode plots.
Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad G(s) = \frac{b_0}{s + a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

**Corner Frequency**: \( \omega = a_0 \)

\[ G(ja_0) = \frac{b_0}{ja_0 + a_0} = \frac{G(0)}{j + 1} \]

\[ G(ja_0) = 0.5G(0) - 0.5G(0)j \]
Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \]

\[ G(s) = \frac{b_0}{s + a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

**Corner Frequency**: \( \omega = a_0 \)

\[ G(ja_0) = \frac{b_0}{j a_0 + a_0} = \frac{G(0)}{j + 1} \]

\[ G(ja_0) = 0.5G(0) - 0.5G(0)j \]

\[ \angle G(ja_0) = -45^\circ \]

\[ |G(ja_0)| = \frac{1}{\sqrt{2}} |G(0)| \]

Time constant is \( \tau = \frac{1}{a_0} \).

Larger corner frequency \( \iff \) Faster Response
Consider the following first-order system:

\[
\dot{y}(t) + a_0 y(t) = b_0 u(t) \quad \quad G(s) = \frac{b_0}{s + a_0}
\]

To start, assume \(a_0 > 0\) and \(b_0 > 0\).

**Low Frequency**: \(\omega \leq \frac{a_0}{10}\)

\[
G(j\omega) \approx \frac{b_0}{a_0}
\]

\[
\angle G(j\omega) = 0^\circ
\]

\[
|G(j\omega)| = G(0)
\]
High-Frequency Approximation

Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad G(s) = \frac{b_0}{s + a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

**High Frequency**: \( \omega \geq 10 a_0 \)

\[
G(j\omega) \approx \frac{b_0}{j\omega} = -\frac{b_0}{\omega} j
\]
High-Frequency Approximation

Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad \quad \quad G(s) = \frac{b_0}{s+a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

**High Frequency**: \( \omega \geq 10a_0 \)

\[ G(j\omega) \approx \frac{b_0}{j\omega} = -\frac{b_0}{\omega} \cdot j \]

\[ \angle G(j\omega) \approx -90^\circ \]

\[ |G(j\omega)| \approx \frac{b_0}{\omega} \]

Magnitude rolls-off at -20dB per decade (similar to \( 1/s \)).
Middle-Frequency Approximation

Consider the following first-order system:

\[ y'(t) + a_0 y(t) = b_0 u(t) \]

\[ G(s) = \frac{b_0}{s+a_0} \]

To start, assume \( a_0 > 0 \) and \( b_0 > 0 \).

**Middle Frequency:**

\[ \frac{a_0}{10} \leq \omega \leq 10a_0 \]

Straight line approximation to connect low/high freqs.

Magnitude: Lines meet at corner frequency.

Phase: Line passes through \(-45^\circ\) at corner frequency.
Consider the following first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \]

\[ G(s) = \frac{b_0}{s+a_0} \]

Allow \( a_0 \) and \( b_0 \) to take any sign.

**Bode Plots:**

- Use same procedure for straight-line approximation.
- Magnitude is unchanged.
- Phase changes by \( \pm 90^\circ \) but details depend on signs of \((a_0, b_0)\).
- Bode plots can be drawn for unstable systems.
First-Order Zero

Consider the following first-order system:

\[ a_0 y(t) = \dot{u}(t) + b_0 u(t) \quad \quad \quad G(s) = \frac{s+b_0}{a_0} \]

Allow \( a_0 \) and \( b_0 \) to take any sign.

**Bode Plots:**

- Use same procedure for straight-line approximation.
- Corner frequency at the zero \( \omega = |a_0| \)
- Magnitude rises at +20dB per decade at high frequencies.
Next, we focus on Bode plots for second order systems.

- Second-order differentiator $G(s) = s^2$: Phase is $+180^\circ$ and magnitude has slope $+40\text{dB/decade}$.

- Second-order integrator $G(s) = \frac{1}{s^2}$: Phase is $-180^\circ$ and magnitude has slope $-40\text{dB/decade}$.

- Second-order underdamped $G(s) = \frac{b_0}{s^2 + 2\zeta \omega_n s + \omega_n^2}$:
  - Magnitude is (approximately) flat up to the corner frequency $\omega_n$ and rolls off at $-40\text{dB/dec}$ at high frequencies.
  - Phase plot transitions by $\pm 180^\circ$ depending on the signs of the coefficients.
  - If damping is low ($\zeta \ll 1$) then the plot has a resonant peak of $|G(j\omega_n)| \approx \left| \frac{G(0)}{2\zeta} \right|$. 

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Bode Plots for 2\textsuperscript{nd}-Order Systems
Bode Plot: Second-Order Differentiator

- Differentiator: \( y(t) = \ddot{u}(t) \)
  - \( G(s) = \frac{s^2}{1} = s^2 \)
  - If \( u(t) = \sin(\omega t) \) then \( y(t) = -\omega^2 \sin(\omega t) = \omega^2 \sin(\omega t + \pi) \)
  - This agrees with \( |G(j\omega)| = \omega^2 \) and \( \angle G(j\omega) = \pi \text{ rad} = +180^\circ \)
- Magnitude has slope +40dB/decade and phase is +180°.

A \( N^{th} \) order differentiator \( G(s) = s^N \) has phase +90\( N \)deg and magnitude slope of +20\( N \)dB per decade.
Bode Plot: Second-Order Integrator

- **Integrator:** 
  \[ \ddot{y}(t) = u(t) \]
  \[ G(s) = \frac{1}{s^2} \]

  - If \( u(t) = \sin(\omega t) \) then 
    \[ y(t) = -\frac{1}{\omega^2} \sin(\omega t) = \frac{1}{\omega^2} \sin(\omega t - \pi) \]
    
    [The form for \( y \) neglects integration constants.]

  - This agrees with \( |G(j\omega)| = \frac{1}{\omega^2} \) and \( \angle G(j\omega) = -\pi \text{ rad} = -180^\circ \)

- **Magnitude** has slope -40dB/decade and **phase** is -180°.

**A N^{th} order integrator**

\[ G(s) = \frac{1}{s^N} \] has phase

-90Ndeg and

magnitude slope of

-20NdB per decade.
Second-Order Underdamped Systems

Consider the a stable, second-order system:
\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

Bode plots can be generated by Matlab:

```matlab
>> b0=16;
>> wn = 4;
>> z1 = 0.7;
>> G1=tf(b0,[1 2*z1*wn wn^2]);
>> z2 = 0.05;
>> G2=tf(b0,[1 2*z2*wn wn^2]);
>> bode(G1,'k-',G2,'b');
```

It will be useful to sketch straight-line approximate Bode plots.

\[ G(s) = \frac{b_0}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ G(s) = \frac{16}{s^2 + 8\zeta s + 16} \]
Consider the a stable, second-order system:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

**Corner Frequency:** \( \omega = \omega_n \)

\[
G(j\omega_n) = \frac{b_0}{(j\omega_n)^2 + 2\zeta \omega_n (j\omega_n) + \omega_n^2}
\]

\[
G(j\omega_n) = \frac{b_0}{2\zeta \omega_n^2 j} = -\frac{G(0)}{2\zeta} j
\]
Corner Frequency

Consider the a stable, second-order system:
\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]
Assume \( b_0 > 0 \).

**Corner Frequency:** \( \omega = \omega_n \)

\[
G(j\omega_n) = \frac{b_0}{(j\omega_n)^2 + 2\zeta \omega_n j + \omega_n^2}
\]

\[
G(j\omega_n) = \frac{b_0}{2\zeta \omega_n^2 j} = -\frac{G(0)}{2\zeta} j
\]

\[ \angle G(j\omega_n) = -90^\circ \]

\[
|G(j\omega_n)| = \frac{1}{2\zeta} |G(0)|
\]

Small \( \zeta \) gives a resonant peak.
This is associated with overshoot and oscillations.
Consider the a stable, second-order system:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

**Low Frequency:** \( \omega \leq \frac{\omega_n}{10} \)

\[ G(j\omega) \approx \frac{b_0}{\omega_n^2} \]

\[ \angle G(j\omega) = 0^o \]

\[ |G(j\omega)| = G(0) \]

\[ G(s) = \frac{b_0}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ G(s) = \frac{16}{s^2 + 8\zeta s + 16} \]
High-Frequency Approximation

Consider the a stable, second-order system:

\[ \ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

**High Frequency**: \( \omega \geq 10\omega_n \)

\[ G(j\omega) \approx \frac{b_0}{(j\omega)^2} = -\frac{b_0}{\omega^2} \]

\[ G(s) = \frac{b_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ G(s) = \frac{16}{s^2 + 8\zeta s + 16} \]
Consider a stable, second-order system:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

**High Frequency:** \( \omega \geq 10\omega_n \)

\[
G(j\omega) \approx \frac{b_0}{(j\omega)^2} = -\frac{b_0}{\omega^2}
\]

\[
\angle G(j\omega) \approx -180^\circ
\]

\[
|G(j\omega)| \approx \frac{b_0}{\omega^2}
\]

Magnitude rolls-off at -40dB per decade (similar to \( 1/s^2 \)).
Consider the a stable, second-order system:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Assume \( b_0 > 0 \).

**Middle Frequency:**

\[ \frac{\omega_n}{10} \leq \omega \leq 10 \omega_n \]

Straight line approximation to connect low/high freqs.

Magnitude: Lines meet at corner frequency.

Phase: Line passes through -90° at corner frequency.

Low \( \zeta \) gives resonant peak and sharp phase transition.
Resonance

• “Lightly” damped second-order system:

\[ G(s) = \frac{16}{s^2 + 8\zeta s + 16} \text{ with } \zeta = 0.05 \]
Resonance

- “Lightly” damped second-order system:

\[ G(s) = \frac{16}{s^2 + 8\zeta s + 16} \text{ with } \zeta = 0.05 \]
General Second-Order Systems

We can draw Bode plots for the following cases using a similar procedure:

\[ G(s) = \frac{b_0}{s^2 \pm 2\zeta \omega_n s \pm \omega_n^2} \quad \text{with} \quad \zeta < 1 \]

\[ G(s) = \frac{s^2 \pm 2\zeta \omega_n s \pm \omega_n^2}{a_0} \quad \text{with} \quad \zeta < 1 \]

Bode plots for higher-order systems are discussed next. The approach can be used to sketch Bode plots for overdamped second-order systems (which can be expressed as a connection of two first-order systems).
Consider a system whose transfer function is \( G(s) = G_1(s)G_2(s) \).

- The Bode phase plot of \( G(s) \) is the sum of the phase plots of \( G_1(s) \) and \( G_2(s) \).
- The Bode magnitude plot of \( G(s) \) (in dB) is the sum of the magnitude plots of \( G_1(s) \) and \( G_2(s) \).

This can be used to draw Bode plots for higher order systems.
Products of Transfer Functions

- Consider a system whose transfer function is $G(s) = G_1(s)G_2(s)$.
- The response of $G(s)$ at frequency $\omega$ is:

$$G(j\omega) = G_1(j\omega)G_2(j\omega) = |G_1(j\omega)|e^{j\angle G_1(j\omega)}|G_2(j\omega)|e^{j\angle G_2(j\omega)}$$

$$\Rightarrow |G(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)| \quad \angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

- Next recall that for any real numbers $c_1$ and $c_2$:

$$\log_{10}(c_1c_2) = \log_{10}(c_1) + \log_{10}(c_2)$$

- Thus the magnitude of $G(j\omega)$ in dB is given by:

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB}$$

The Bode phase plot of $G(s)$ is the sum of the phase plots of $G_1(s)$ and $G_2(s)$. The Bode magnitude plot of $G(s)$ (in dB) is the sum of the magnitude plots of $G_1(s)$ and $G_2(s)$. 
Example: Lead Controller

• Consider the first-order system:
  \[ \dot{u}(t) + 2u(t) = 2\dot{e}(t) + e(t) \quad G(s) = \frac{2s+1}{s+2} \]

• Express transfer function as a product:
  \[ G(s) = G_1(s)G_2(s) \text{ where } G_1(s) = 2s + 1 \text{ and } G_2(s) = \frac{1}{s+2}. \]
Example: Overdamped Second-Order System

- Consider the first-order system:
  \[
  \ddot{y}(t) + 1.2\dot{y}(t) + 0.2y(t) = 0.5u(t)
  \]
  \[
  G(s) = \frac{0.5}{s^2 + 1.2s + 0.2}
  \]

- Express transfer function as a product:
  \[
  G(s) = G_1(s)G_2(s) \quad \text{where} \quad G_1(s) = \frac{1}{s+0.2} \quad \text{and} \quad G_2(s) = \frac{0.5}{s+1}.
  \]

Bode Diagram

- Frequency (rad/sec) vs. Magnitude (dB)
- Frequency (rad/sec) vs. Phase (deg)