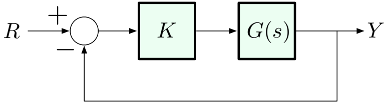
Plan of the Lecture

- Review: control design using frequency response: PI/lead
- Today's topic: control design using frequency response: PD/lag, PID/lead+lag

Goal: understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot; develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

Reading: FPE, Chapter 6

Review: Bode's Gain-Phase Relationship



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

	low freq.	real zero/pole	complex zero/pole
mag. slope	n	up/down by 1	up/down by 2
phase	$n \times 90°$	up/down by 90°	up/down by 180°

We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

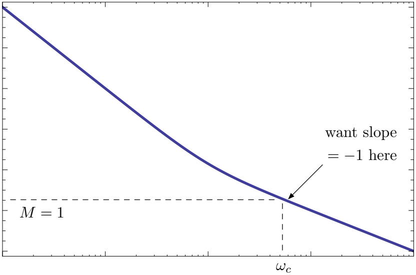
$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

Bode's Gain-Phase Relationship

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$

This suggests the following rule of thumb:

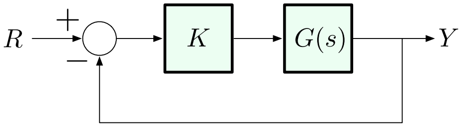


- M has slope -2 at ω_c
  $\Rightarrow \phi(\omega_c) = -180^\circ$
  $\Rightarrow$ bad (no PM)
- M has slope -1 at ω_c
  $\Rightarrow \phi(\omega_c) = -90^\circ$
  $\Rightarrow$ good (PM $= 90^\circ$)

— this is an important *design guideline*!!

(Similar considerations apply when M-plot has positive slope – depends on the t.f.)

Control Design Using Frequency Response



Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing K (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \qquad \Longrightarrow \qquad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of 90° and consequently good damping.

Lead Controller Design Using Frequency Response

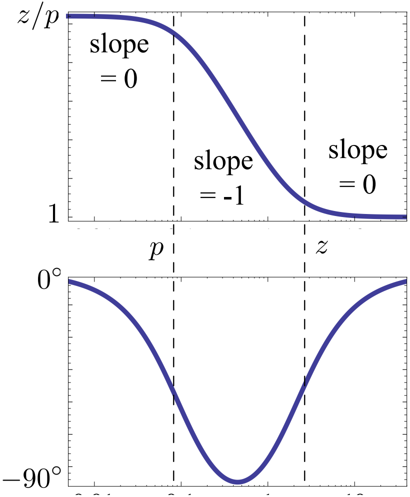
General Procedure

1. Choose K to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   - in general, we should first check PM with the K from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it's not very precise, requires trial & error.

Lag Compensation: Bode Plot

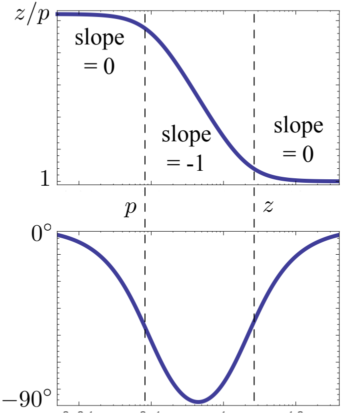
$$D(s) = \frac{s+z}{s+p} = \frac{z}{p}\frac{\frac{s}{z}+1}{\frac{s}{p}+1}, \qquad z \gg p$$



- $\frac{j\omega + z}{j\omega + p} \xrightarrow{\omega\to\infty} 1$
  so $M \to 1$ at high frequencies

- subtracts phase, hence the term "phase lag"

Lag Compensation: Bode Plot



- $\dfrac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to 0} \dfrac{z}{p}$

  steady-state tracking error:

$$e(\infty) = \frac{sR(s)}{1 + D(s)G(s)}\bigg|_{s=0}$$

  large $z/p \implies$ better s.s. tracking

- lag decreases $\omega_c \implies$ slows down time response (to compensate, adjust K or add lead)
- caution: lead increases PM, but adding lag can undo this
- to mitigate this, choose both z and p very small, while maintaining desired ratio z/p

Example

$$G(s) = \frac{1}{(s+0.2)(s+0.5)} \overset{\text{Bode form}}{=} \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

Objectives:

- PM $\geq 60^\circ$
- $e(\infty) \leq 10\%$ for constant reference (closed-loop tracking error)

Strategy:

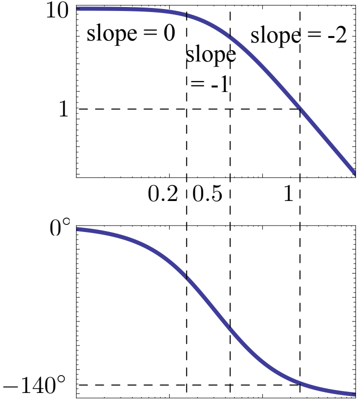
- we will use lag

$$KD(s) = K\frac{s+z}{s+p}, \qquad z \gg p$$

- z and p will be chosen to get good tracking
- PM will be shaped by choosing K
- this is different from what we did for lead (used p and z to shape PM, then chose K to get desired bandwidth spec)

Step 1: Choose K to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:



- from Matlab, $\omega_c \approx 1$
- PM $\approx 40^\circ$
- we want PM $= 60^\circ$

$$\phi = -120^\circ \quad \text{at } \omega \approx 0.573$$
$$M = 2.16$$

— need to decrease K to $1/2.16$

A conservative choice (to allow some slack) is $K = 1/2.5 = 0.4$, gives $\omega_c \approx 0.52$, PM $\approx 65^\circ$

Step 2: Choose z & p to Shape Tracking Error

So far: $KG(s) = \frac{0.4 \cdot 10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$

$$e(\infty) = \frac{1}{1+KG(s)}\bigg|_{s=0} = \frac{1}{1+4} = \frac{1}{5} = 20\% \quad \text{(too high)}$$

To have $e(\infty) \le 10\%$, need $KD(0)G(0) \ge 9$:

$$e(\infty) = \frac{1}{1+KD(0)G(0)} \le \frac{1}{1+9} = 10\%.$$

So, we need

$$D(0) = \frac{s+z}{s+p}\bigg|_{s=0} = \frac{z}{p} \ge \frac{9}{4} = 2.25 \qquad \text{— say, } z/p = 2.5$$

Not to distort PM and ω_c, let's pick z and p an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05,\ p = 0.02$

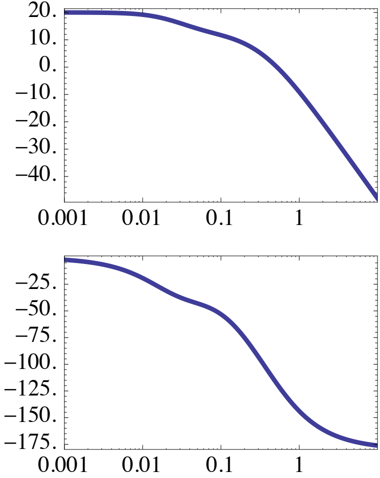
Overall Design

Plant:

$$G(s) = \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

Controller:

$$KD(s) = 0.4\frac{s+0.05}{s+0.02}$$

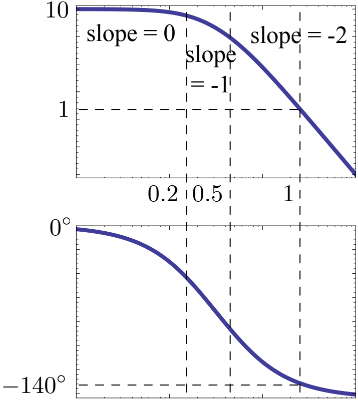


— the design still needs a bit of refinement ...

Lead & Lag Compensation

Let's combine the advantages of PD/lead and PI/lag.

Back to our example: $G(s) = \dfrac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$



- from Matlab, $\omega_c \approx 1$
- PM $\approx 40°$

New objectives:

- $\omega_{\mathrm{BW}} \geq 2$
- PM $\geq 60°$
- $e(\infty) \leq 1\%$ for const. ref.

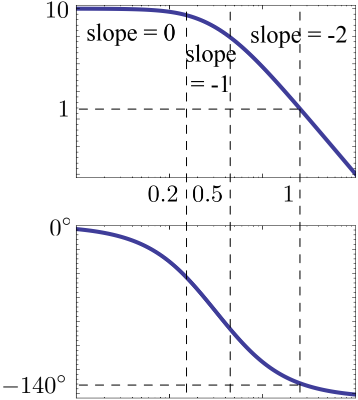
Lead & Lag Compensation

What we got before, with lag only:

- Improved PM by adjusting K to decrease ω_c.
- This gave $\omega_c \approx 0.5$, whereas now we want a larger ω_c (recall: $\omega_{\mathrm{BW}} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.

Lead & Lag Compensation



Step 1. Choose K to get $\omega_c \approx 2$ (before lead)

Using Matlab, can check:

at $\omega = 2, \qquad M \approx 0.24$ (with $K = 1$)

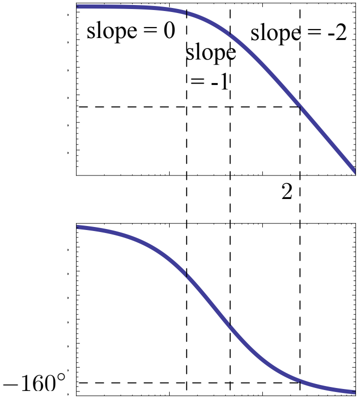
— need $K = \dfrac{1}{0.24} \approx 4.1667$

— choose $K = 4$

(gives ω_c slightly < 2, but still ok).

Lead & Lag Compensation

$K = 4$



Step 2. Decide how much phase lead is needed, and choose z_{lead} and p_{lead}

Using Matlab, can check:

$$\text{at } \omega = 2, \qquad \phi \approx -160^\circ$$

— so PM = 20°

(in fact, choosing $K = 4$ made things worse: it increased ω_c and consequently decreased PM)

We need at least 40° phase lead!!

The choice of lead pole/zero must satisfy

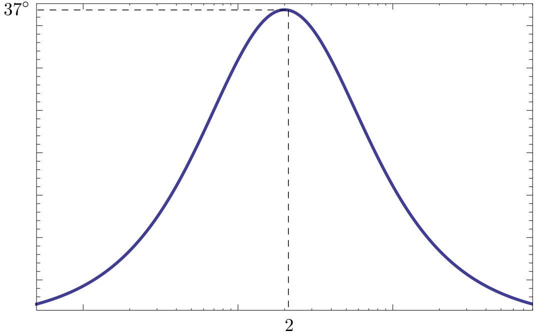
$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Let's try $z_{\text{lead}} = 1$ and $p_{\text{lead}} = 4$ $\qquad D(s) = \dfrac{s+1}{\frac{s}{4}+1}$



Phase lead = 37° — not enough!!

Lead & Lag Compensation

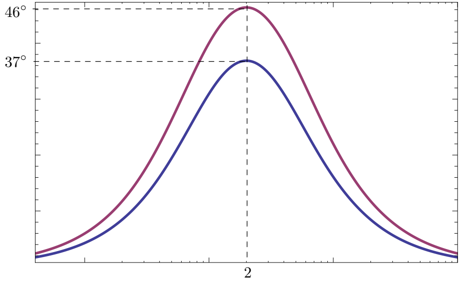
Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

The choice of $z_{\text{lead}} = 1$, $p_{\text{lead}} = 4$ gave phase lead $= 37°$.

Need to space z_{lead} and p_{lead} farther apart:

$$\begin{cases} z_{\text{lead}} = 0.8 \\ p_{\text{lead}} = 5 \end{cases} \implies \text{phase lead } = 46°$$



Lead & Lag Compensation

Step 3. Evaluate steady-state tracking and choose $z_{\text{lag}}, p_{\text{lag}}$ to satisfy specs

So far:

$$K \underbrace{D(s)}_{\substack{\text{lead}\\\text{only}}} G(s) = 4 \frac{\frac{s}{0.8}+1}{\frac{s}{5}+1} \cdot \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

$$KD(0)G(0) = 40 \quad \Longrightarrow \quad e(\infty) = \frac{1}{1+KD(0)G(0)} = \frac{1}{1+40}$$

— this is not small enough: need $1\% = \frac{1}{100} = \frac{1}{1+99}$

We want $D(0) \geq \frac{99}{40}$ with lag $\quad \frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$ will do

Lead & Lag Compensation

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$.

We can stick with our previous design:

$$z_{\text{lag}} = 0.05, \qquad p_{\text{lag}} = 0.02$$

Overall controller:

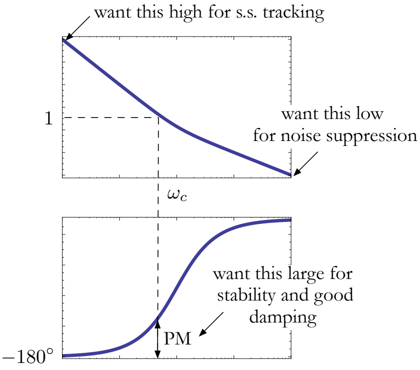
$$\underbrace{4\frac{\frac{s}{0.8}+1}{\frac{s}{5}+1}}_{\substack{\text{lead (with}\\ \text{gain } K=4 \text{ absorbed)}}} \cdot \underbrace{\frac{s+0.05}{s+0.02}}_{\substack{\text{lag (not in}\\ \text{Bode form)}}}$$

(Note: we don't rewrite lag in Bode form, because $z_{\text{lag}}/p_{\text{lag}}$ is not incorporated into K.)

Frequency Domain Design Method: Advantages

Design based on Bode plots is good for:

- easily visualizing the concepts



- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)

Frequency Domain Design Method: Disadvantages

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given K is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given K is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the next lecture.