Plan of the Lecture

- Review: control, feedback, etc.
- Today's topic: linear systems and their dynamic response

Goal: a general framework that encompasses all examples of interest. Once we have mastered this framework, we can proceed to *analysis* and then to *design*.

Notation Reminder

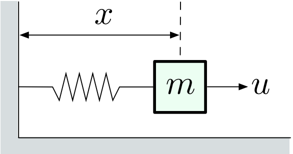
We will be looking at *dynamic systems* whose evolution *in time* is described by *differential equations* with *external inputs*.

We will not write the time variable t explicitly, so we use

$$
\begin{array}{lll}
x & \text{instead of} & x(t) \\
\dot{x} & \text{instead of} & x'(t) \text{ or } \dfrac{\mathrm{d}x}{\mathrm{d}t} \\
\ddot{x} & \text{instead of} & x''(t) \text{ or } \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}
\end{array}
$$

etc.

Example 1: Mass-Spring System



Newton's second law (translational motion):

$$\underbrace{F}_{\text{total force}} = ma = \text{spring force} + \text{friction} + \text{external force}$$

$$\text{spring force} = -kx \quad \text{(Hooke's law)}$$

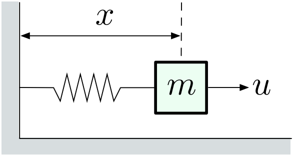
$$\text{friction force} = -\rho\dot{x} \quad \text{(Stokes' law — linear drag, only an approximation!!)}$$

$$m\ddot{x} = -kx - \rho\dot{x} + u$$

Move $x, \dot{x}, \ddot{x}$ to the LHS, u to the RHS:

$$\ddot{x} + \frac{\rho}{m}\dot{x} + \frac{k}{m}x = \frac{u}{m} \qquad \text{2nd-order linear ODE}$$

Example 1: Mass-Spring System



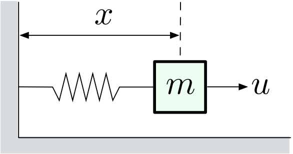
$$\ddot{x} + \frac{\rho}{m}\dot{x} + \frac{k}{m}x = \frac{u}{m}$$ 2nd-order linear ODE

Canonical form: convert to a *system of 1st-order ODEs*

$$\dot{x} = v \qquad \text{(definition of velocity)}$$

$$\dot{v} = -\frac{\rho}{m}v - \frac{k}{m}x + \frac{1}{m}u$$

Example 1: Mass-Spring System



State-space model: express in *matrix form*

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$

Important: start reviewing your linear algebra ***now***!!

- matrix-vector multiplication; eigenvalues and eigenvectors; etc.

General n-Dimensional State-Space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A \\ \\ \underset{\text{matrix}}{n \times n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} B \\ \\ \underset{\text{matrix}}{n \times m} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

$$\dot{x} = Ax + Bu$$

Partial Measurements

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

$$\text{output } y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p \qquad y = Cx \qquad C - p \times n \text{ matrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Example: if we only care about (or can only measure) x_1, then

$$y = x_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

State-Space Models: Bottom Line

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State-space models are useful and convenient for writing down system models for different types of systems, in a unified manner.

When working with state-space models, what are *states* and what are *inputs*?

— match against $\dot{x} = Ax + Bu$

State-Space Models

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where:

- $x(t) \in \mathbb{R}^n$ is the state at time t
- $u(t) \in \mathbb{R}^m$ is the input at time t
- $y(t) \in \mathbb{R}^p$ is the output at time t

and

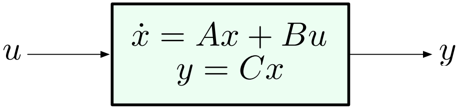
- $A \in \mathbb{R}^{n \times n}$ is the dynamics matrix
- $B \in \mathbb{R}^{n \times m}$ is the control matrix
- $C \in \mathbb{R}^{p \times n}$ is the sensor matrix

How do we determine the output y for a given input u?

Reminder: we will only consider single-input, single-output (SISO) systems, i.e., $u(t), y(t) \in \mathbb{R}$ for all times t of interest. $(m = p = 1)$

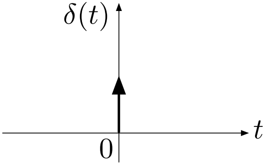
Impulse Response

(Review from ECE 210)

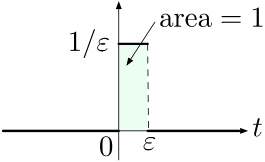


Unit impulse (or Dirac's δ-function):

1. $\delta(t) = 0$ for all $t \neq 0$
2. $\int_{-a}^{a} \delta(t) \mathrm{d}t = 1$ for all $a > 0$

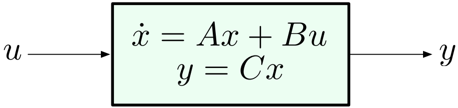


It is useful to think of $\delta(t)$ as a limit of impulses of unit area:



as $\varepsilon \to 0$, the impulse gets taller ($1/\varepsilon \to +\infty$), but the area under its graph remains at 1

Impulse Response



zero initial condition: $x(0) = 0$

Consider the input

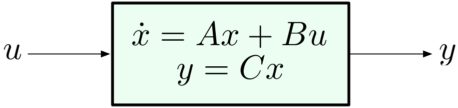
$$u(t) = \delta(t - \tau) \qquad \text{unit impulse applied at } t = \tau$$

The system is *linear* and *time-invariant* (LTI), with zero I.C.:

$$u(t) = \delta(t - \tau) \quad \xrightarrow{x(0)=0;\ \text{LTI system}} \quad y(t) = h(t - \tau)$$

The function h is the impulse response of the system.

Impulse Response



zero initial condition: $x(0) = 0$

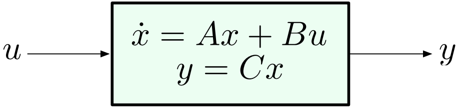
$$u(t) = \delta(t-\tau) \xrightarrow{x(0)=0;\ \text{LTI system}} y(t) = h(t-\tau)$$

Questions to consider:

1. If we know h, how can we find the system's response to other (arbitrary) inputs?
2. If we don't know h, how can we determine it?

We will start with Question 1.

Impulse Response



zero initial condition: $x(0) = 0$

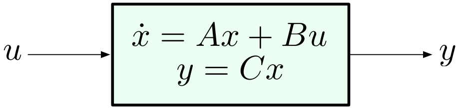
Question: If we know h, how can we find the system's response to other (arbitrary) inputs?

Recall the *sifting property* of the δ-function: for any function f which is "well-behaved" at $t = \tau$,

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\mathrm{d}t = f(\tau)$$

— any *reasonably regular* function can be represented as an integral of impulses!!

Impulse Response



zero initial condition: $x(0) = 0$

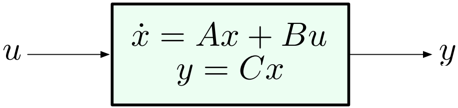
Question: If we know h, how can we find the system's response to other (arbitrary) inputs?

By the sifting property, for a general input $u(t)$ we can write

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\mathrm{d}\tau.$$

Now we recall the *superposition principle:* the response of a linear system to a sum (or integral) of inputs is the sum (or integral) of the individual responses to these inputs.

Impulse Response



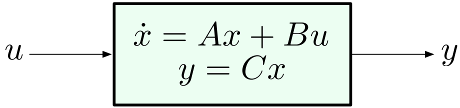
zero initial condition: $x(0) = 0$

The *superposition principle:* the response of a linear system to a sum (or integral) of inputs is the sum (or integral) of the individual responses to these inputs.

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\mathrm{d}\tau \quad \longrightarrow \quad y(t) = \int_{-\infty}^{\infty} u(\tau) \underbrace{h(t-\tau)}_{\substack{\text{response to} \\ \delta(t-\tau)}} \mathrm{d}\tau$$

— the integral that defines $y(t)$ is a convolution of u and h.

Impulse Response



zero initial condition: $x(0) = 0$

Conclusion so far: for zero initial conditions, the output is the convolution of the input with the system impulse response:

$$y(t) = u(t) \star h(t) = h(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

Q: Does this formula provide a *practical* way of computing the output y for a given input u?

A: Not directly (computing convolutions is not exactly pleasant), but ...we can use Laplace transforms.

Laplace Transforms and the Transfer Function

Reminder: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau) e^{-s\tau} \mathrm{d}\tau, \qquad s \in \mathbb{C}$$

time domain	frequency domain
$u(t)$	$U(s)$
$h(t)$	$H(s)$
$y(t)$	$Y(s)$

convolution in time domain $\longleftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \longleftrightarrow \quad Y(s) = H(s)U(s)$$

The Laplace transform of the impulse response

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \mathrm{d}\tau$$

is called the transfer function of the system.

Laplace Transforms and the Transfer Function

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Limits of integration:

- We only deal with *causal* systems — output at time t is not affected by inputs at future times $t' > t$
- If the system is causal, then $h(t) = 0$ for $t < 0$ — $h(t)$ is the response at time t to a unit impulse at time 0
- We will take all other possible inputs (not just impulses) to be 0 for $t < 0$, and work with *one-sided* Laplace transforms:

$$y(t) = \int_{0}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

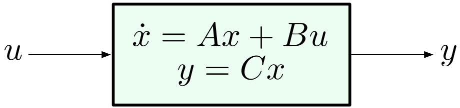
$$H(s) = \int_{0}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Laplace Transforms and the Transfer Function

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Given $u(t)$, we can find $U(s)$ using tables of Laplace transforms or `MATLAB`. But how do we know $h(t)$ [or $H(s)$]?

- Suppose we have a state-space model:



In this case, we have an exact formula:

$$H(s) = C(Is - A)^{-1}B \qquad \text{(matrix inversion)}$$

$$h(t) = Ce^{At}B,\ t \geq 0^{-} \qquad \text{(matrix exponential)}$$

— will not encounter this until much later in the semester.

Laplace Transforms and the Transfer Function

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

- So, how should we compute $H(s)$ in practice?

  Try injecting some specific inputs and see what happens at the output.

  Let's try $u(t) = e^{st}, t \geq 0$ (s is some fixed number)

$$\begin{aligned} y(t) &= \int_0^{\infty} h(\tau)u(t-\tau)\mathrm{d}\tau \qquad \text{(because } u \star h = h \star u\text{)} \\ &= \int_0^{\infty} h(\tau)e^{s(t-\tau)}\mathrm{d}\tau \\ &= e^{st}\int_0^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau \\ &= e^{st}H(s) \end{aligned}$$

– so, $u(t) = e^{st}$ is multiplied by $H(s)$ to give the output.

Example

$$\dot{y} = -ay + u \qquad \text{(think } y = x\text{, full measurement)}$$
$$u(t) = e^{st} \qquad \text{(always assume } u(t) = 0 \text{ for } t < 0\text{)}$$
$$y(t) = H(s)e^{st} \qquad \text{— what is } H\text{?}$$

Let's use the system model:

$$\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(H(s)e^{st}\right) = sH(s)e^{st}$$

Substitute into $\dot{y} = -ay + u$:

$$sH(s)\cancel{e^{st}} = -aH(s)\cancel{e^{st}} + \cancel{e^{st}} \qquad (\forall s; t > 0)$$
$$sH(s) = -aH(s) + 1$$

$$H(s) = \frac{1}{s+a} \qquad \Longrightarrow \qquad y(t) = \frac{e^{st}}{s+a}$$

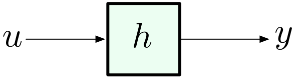
Example (continued)

$$\dot{y} = -ay + u$$
$$H(s) = \frac{1}{s+a}$$

Now we can fund the impulse response $h(t)$ by taking the inverse Laplace transform — from tables,

$$h(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Determining the Impulse Response



$$u(t) = e^{st},\ t \geq 0 \quad \xrightarrow{x(0)=0;\ \text{LTI system}} \quad y(t) = e^{st} H(s)$$

Back to our two questions:

1. If we know h, how can we find y for a given u?
2. If we don't know h, how can we determine it?

We have answered Question 1. Now let's turn to Question 2.

One idea: inject the input $u(t) = e^{st}$, determine $y(t)$, compute

$$H(s) = \frac{y(t)}{u(t)};$$

repeat for all s of interest. Q: Is this a good idea?

Determining the Impulse Response

$$u(t) = e^{st} \longrightarrow \boxed{h} \longrightarrow y(t) = e^{st}H(s)$$

compute $H(s) = \dfrac{y(t)}{u(t)}$, repeat for as many values of s as necessary

Q: Is this likely to work *in practice*?

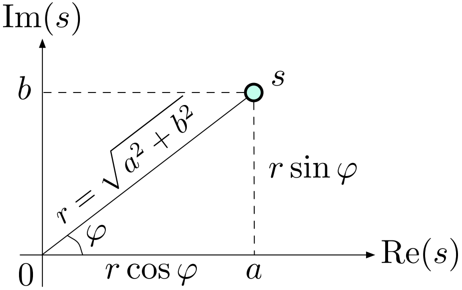
A: No — e^{st} blows up very quickly if $s > 0$, and decays to 0 very quickly if $s < 0$.

So we need *sustained, bounded signals* as inputs.

This is possible if we allow s to take on *complex values*.

Review: Complex Numbers

$$s = \underbrace{a}_{\text{real part}} + j \underbrace{b}_{\text{imaginary part}}$$ — rectangular form



Polar form:

$$s = re^{j\varphi}$$

$$r = |s| = \sqrt{a^2 + b^2}$$

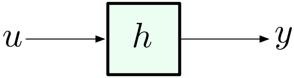
(magnitude)

$$\varphi = \angle s = \tan^{-1}\left(\frac{b}{a}\right)$$

(phase)

Euler's formula: $e^{j\varphi} = \cos\varphi + j\sin\varphi$

Frequency Response



$u(t) = A\cos(\omega t)$ $\qquad$ A – amplitude; ω – (angular) frequency, rad/s

From Euler's formula:

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

By linearity, the response is

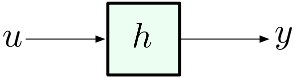
$$y(t) = \frac{A}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

$$\text{where } H(j\omega) = \int_0^\infty h(\tau)e^{-j\omega\tau}\mathrm{d}\tau$$

$$H(-j\omega) = \int_0^\infty \underbrace{h(\tau)e^{j\omega\tau}}_{\substack{\text{complex}\\ \text{conjugate}}}\mathrm{d}\tau = \overline{H(j\omega)}$$

(recall that $h(\tau)$ is real-valued)

Frequency Response



$$u(t) = A\cos(\omega t) \quad \longrightarrow \quad y(t) = \frac{A}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

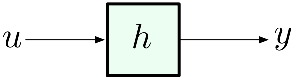
$$H(j\omega) \in \mathbb{C} \quad \Longrightarrow \quad \begin{aligned} H(j\omega) &= M(\omega)e^{j\varphi(\omega)} \\ H(-j\omega) &= M(\omega)e^{-j\varphi(\omega)} \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \frac{A}{2}M(\omega)\Big[e^{j(\omega t + \varphi(\omega))} + e^{-j(\omega t + \varphi(\omega))}\Big] \\ &= AM(\omega)\cos\big(\omega t + \varphi(\omega)\big) \quad \text{(only true in \textit{steady state})} \end{aligned}$$

The (steady-state) response to a cosine signal with amplitude A and frequency ω is still a cosine signal with amplitude $AM(\omega)$, same frequency ω, and phase shift $\varphi(\omega)$

Frequency Response



$$u(t) = A\cos(\omega t) \quad \longrightarrow \quad y(t) = A \underbrace{M(\omega)}_{\substack{\text{amplitude}\\ \text{magnification}}} \cos\big(\omega t + \underbrace{\varphi(\omega)}_{\substack{\text{phase}\\ \text{shift}}}\big)$$

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)? — always given by $Y(s) = H(s)U(s)$
- What about response under *nonzero I.C.'s*?— we will see that, if *the system is stable*, then

$$\text{total response} = \begin{array}{c}\text{transient response}\\ \text{(depends on I.C.)}\end{array} + \begin{array}{c}\text{steady-state response}\\ \text{(independent of I.C.)}\end{array}$$

— need more on Laplace transforms