ECE 486: Control Systems

Lecture 3A: Response of Linear ODEs

Key Takeaways

This lecture focuses on exact, analytical solution of the free and forced response of a linear ODE with constant coefficients.

The lecture covers the following:

1. Basic terminology: Poles, zeros, and DC (steady-state) gain
2. Minimal realizations
3. Form of the general free response solution
4. Form of the general forced response solution

The properties of the general solutions are used to understand the impact of feedback control on the closed-loop dynamics.

Terminology

Consider a transfer function for an nth-order LTI system:

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

- A number $p \in C$ is a pole or root of G(s) if it is a solution of the characteristic equation:

$$a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0$$

- A number $z \in C$ is a zero of G(s) if it is a solution of:

$$b_m z^m + \cdots + b_1 z + b_0 = 0.$$

- The DC gain or steady-state gain is $G(0) = \frac{b_0}{a_0}$

Typically *G(z)=0* at a zero *z* and *G(p)=∞* at some pole *p*.

Matlab Example

$$\ddot{y}(t) + 10\dot{y}(t) + 169y(t) = 3042\dot{u}(t) + 1014u(t)$$

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>> G = tf([3042 1014],[1 10 169]);
>> zero(G) % Zeros are roots of 3042s+1014=0
 ans = -0.3333
>> pole(G) % Poles are roots of s^2+10s+169=0
 ans =
  -5.0000 +12.0000i
  -5.0000 -12.0000i
>> dcgain(G) % DC gain G(0) = 1014/169
 ans =6
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Minimal Realizations

- A system is called non-minimal if it has a pole and zero at the same location. It is called minimal otherwise.
- Example 1: Not Minimal

$$\dot{y}(t) + y(t) = 2\dot{u}(t) + 2u(t)$$

$$G(s) = \frac{2s+2}{s+1} = \frac{2(s+1)}{s+1}$$

$G(-1)$ is not well-defined.

- Example 2: Minimal

$$\dot{y}(t) + 3y(t) = 2\dot{u}(t) + 8u(t)$$

$$G(s) = \frac{2s+8}{s+3} = \frac{2(s+4)}{s+3}$$

Non-minimal systems "hide" some dynamics from the input/output relation. Example 1 is first-order but the input-output relation is equivalent (with zero ICs) to *y=2u*.

Homogeneous Solutions

We want to characterize the free/forced responses.

First, we find homogeneous solutions. These are solutions to the unforced ODE (neglecting the ICs for the moment):

$$a_3 y^{[3]}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0$$

"Guess" that $y(t) = e^{st}$ is a solution for some $s \in C$.

Note that: $\dot{y}(t) = se^{st}, \quad \ddot{y}(t) = s^2 e^{st}, \quad y^{[3]}(t) = s^3 e^{st},$

Substitute into the ODE: $\left(a_3 s^3 + a_2 s^2 + a_1 s + a_0\right) e^{st} = 0$

Thus $y(t) = e^{st}$ is a homogeneous solution if and only if s solves the characteristic equation:

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

Free Response Solution

Every homogeneous solution of

$$a_3 y^{[3]}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0$$

has the form:

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_3 e^{s_3 t}$$

where $\{c_1, c_2, c_3\}$ are constants and $\{s_1, s_2, s_3\}$ are roots of the characteristic equation:

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

The roots can be real and/or complex.

Free (Initial Condition) Response

Consider the n^{th} order ODE with initial conditions:

$$a_3 y^{[3]}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0$$

$$\text{ICs: } \; y(0) = y_0; \; \dot{y}(0) = y_0^{[1]}; \; \ddot{y}(0) = y_0^{[2]}$$

The free response is obtained by:

1. Solving for the roots $\{s_1, s_2, s_3\}$
2. Forming the general homogeneous solution:

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_3 e^{s_3 t}$$

3. Using the 3 initial conditions to solve for $\{c_1, c_2, c_3\}$.

[In general there are *n* equations and *n* unknowns.]

Forced Response

Consider the n^{th} order ODE with initial conditions:

$$a_3 y^{[3]}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$$

$$\text{ICs: } \; y(0) = y_0; \; \dot{y}(0) = y_0^{[1]}; \; \ddot{y}(0) = y_0^{[2]}$$

The forced response is obtained by:

1. Solving for the roots $\{s_1, s_2, s_3\}$
2. Finding any particular solution y_P that solves the ODE with the forcing (but not necessarily the ICs)

Example: If $u(t) = \bar{u}$ (constant) then $y_P(t) = \frac{b_0}{a_0} \bar{u}$ (constant) is a particular solution.

Forced Response

Consider the n^{th} order ODE with initial conditions:

$$a_3 y^{[3]}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$$

$$\text{ICs:} \;\; y(0) = y_0;\; \dot{y}(0) = y_0^{[1]};\; \ddot{y}(0) = y_0^{[2]}$$

The forced response is obtained by:

1. Solving for the roots $\{s_1, s_2, s_3\}$
2. Finding any particular solution y_P that solves the ODE with the forcing (but not necessarily the ICs)
3. Forming the general solution:

$$y(t) = y_P(t) + c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_3 e^{s_3 t}$$

4. Using the 3 initial conditions to solve for $\{c_1, c_2, c_3\}$.

Complex Roots

The characteristic equation roots may be complex:

$$s_1 = \alpha + j\beta \text{ where } j = \sqrt{-1} \text{ and } \alpha, \beta \text{ are real}$$

Any complex roots come in complex conjugate pairs:

$$s_1 = \alpha + j\beta \text{ and } s_2 = \alpha - j\beta$$

This leads to complex exponential terms in the solutions:

$$c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

These terms can be re-written using Euler's formula as:

$$c_1 e^{s_1 t} + c_2 e^{s_2 t} = \tilde{c}_1 e^{\alpha t} \cos(\beta t) + \hat{c}_1 e^{\alpha t} \sin(\beta t)$$

where $\tilde{c}_1$ and $\hat{c}_1$ are (new) real coefficients.