ECE 486: Control Systems

- Lecture 17B: Bode's Gain-Phase Relationship

Goal: understand Bode's gain-phase relationship and its importance for control design

Reading: FPE, Chapter 6

Review: Phase Margin for 2nd-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}, \qquad \text{closed-loop t.f.} \; = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

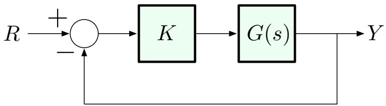
$$\text{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right) \approx 100 \cdot \zeta$$

Conclusions:

larger PM	$\Longleftrightarrow$	better damping
(open-loop quantity)		(closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Bode's Gain-Phase Relationship



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

	low freq.	real zero/pole	complex zero/pole
mag. slope	n	up/down by 1	up/down by 2
phase	$n \times 90°$	up/down by 90°	up/down by 180°

We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

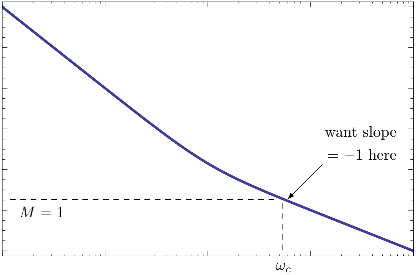
$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

Bode's Gain-Phase Relationship

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$

This suggests the following rule of thumb:

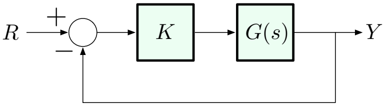


- M has slope -2 at ω_c $\Rightarrow \phi(\omega_c) = -180^\circ$ $\Rightarrow$ bad (no PM)
- M has slope -1 at ω_c $\Rightarrow \phi(\omega_c) = -90^\circ$ $\Rightarrow$ good ($\text{PM} = 90^\circ$)

— this is an important *design guideline*!!

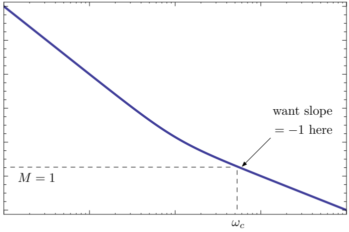
(Similar considerations apply when M-plot has positive slope – depends on the t.f.)

Gain-Phase Relationship & Bandwidth



$$\begin{cases} |KG(j\omega_c)| = 1 \\ \angle G(j\omega_c) = -90^\circ \end{cases} \quad \Rightarrow KG(j\omega_c) = -j$$

M-plot for *open-loop* t.f. KG:



Note: $|KG(j\omega)| \to \infty$ as $\omega \to 0$

Closed-loop t.f.:

$$T(j\omega_c) = \frac{KG(j\omega_c)}{1 + KG(j\omega_c)} = \frac{-j}{1-j}$$

$$|T(j\omega_c)| = \left|\frac{-j}{1-j}\right| = \frac{1}{\sqrt{2}}$$

$$|T(0)| = \lim_{\omega \to 0} \frac{|KG(j\omega)|}{|1 + KG(j\omega)|} = 1$$

$$\Longrightarrow \omega_c = \omega_{\text{BW}} \text{ (bandwidth)}$$

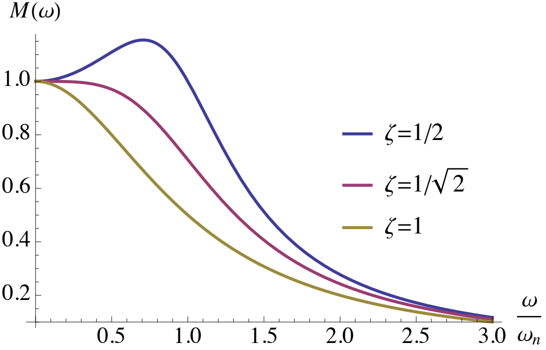
- If PM $= 90^\circ$, then $\omega_c = \omega_{\text{BW}}$
- If $PM < 90^\circ$, then $\omega_c \le \omega_{\text{BW}} \le 2\omega_c$ (see FPE)

Bandwidth

For our prototype 2nd-order system:

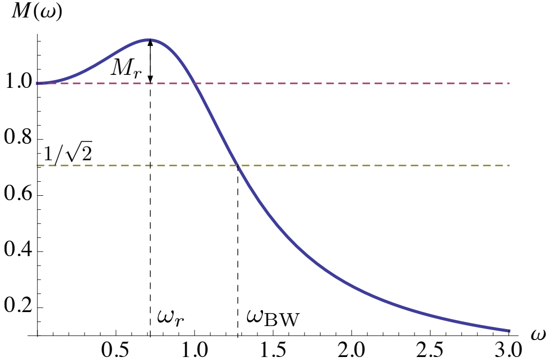
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1 + (4\zeta^2 - 2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4}}$$



Bandwidth

Here is a typical frequency response magnitude plot:

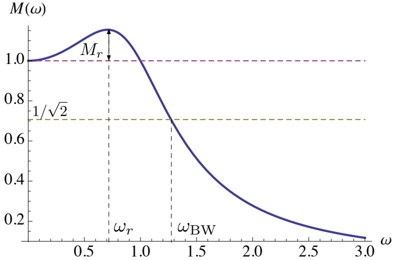


ω_r – resonant frequency

M_r – resonant peak

$\omega_{\rm BW}$ – bandwidth

Bandwidth



We can get the following formulas using calculus:

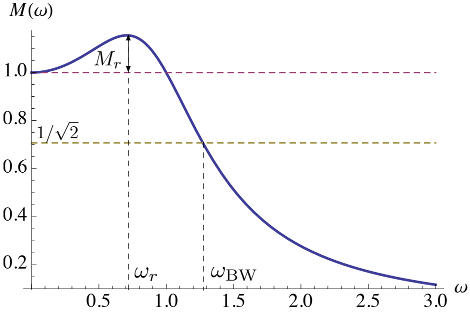
$$\begin{cases} \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \qquad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}};\ \text{for } \zeta \ge \frac{1}{\sqrt{2}},\ \omega_r = 0\right)$$

$$\omega_{\mathrm{BW}} = \omega_n \underbrace{\sqrt{(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2 + 1}}}_{=1 \text{ for } \zeta = 1/\sqrt{2}}$$

— so, if we know $\omega_r, M_r, \omega_{\mathrm{BW}}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Bandwidth

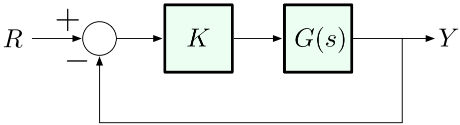
All information about time response is also encoded in frequency response!!



$$\text{small } M_r \longleftrightarrow \text{better damping}$$

$$\text{large } \omega_{\rm BW} \longleftrightarrow \text{large } \omega_n \longleftrightarrow \text{smaller } t_r$$

Control Design Using Frequency Response



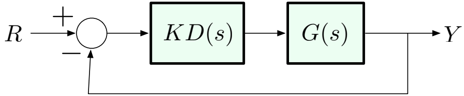
Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing K (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \qquad \Longrightarrow \qquad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of 90° and consequently good damping.

Example



$$\text{Let } G(s) = \frac{1}{s^2} \qquad \text{(double integrator)}$$

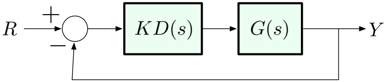
Objective: design a controller $KD(s)$ (K = scalar gain) to give

- stability
- good damping (will make this more precise in a bit)
- $\omega_{\text{BW}} \approx 0.5$ (always a closed-loop characteristic)

Strategy:

- from Bode's Gain-Phase Relationship, we want magnitude slope $= -1$ at $\omega_c \Longrightarrow \text{PM} = 90^\circ \Longrightarrow$ good damping;
- if $\text{PM} = 90^\circ$, then $\omega_c = \omega_{\text{BW}} \Longrightarrow$ want $\omega_c \approx 0.5$

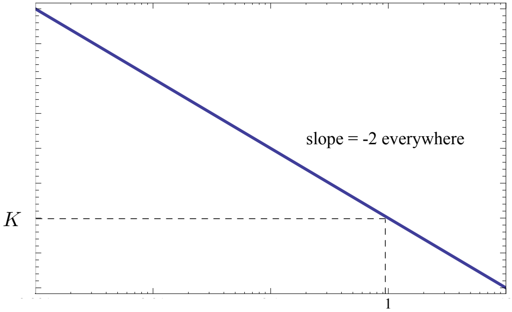
Design, First Attempt



$$G(s) = \frac{1}{s^2}$$

Let's try proportional feedback:

$$D(s) = 1 \implies KD(s)G(s) = KG(s) = \frac{K}{s^2}$$

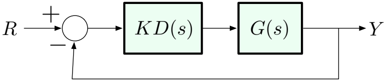


This is not a good idea: slope $= -2$ everywhere, so no PM.

We already know that P-gain alone won't do the job:

$$K + s^2 = 0 \text{ (imag. poles)}$$

Design, Second Attempt



$$G(s) = \frac{1}{s^2}$$

Let's try proportional-derivative feedback:

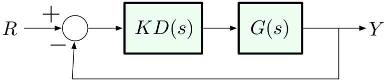
$$KD(s) = K(\tau s + 1), \qquad \text{where } K = K_{\mathrm{P}},\ K\tau = K_{\mathrm{D}}$$

Open-loop transfer function: $KD(s)G(s) = \dfrac{K(\tau s + 1)}{s^2}$.

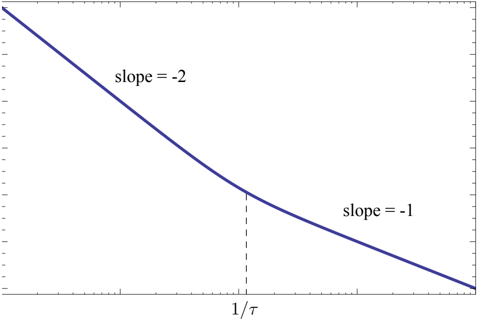
Bode plot interpretation: PD controller introduces a Type 2 term in the numerator, which pushes the slope up by 1

— this has the effect of pushing the M-slope of $KD(s)G(s)$ from -2 to -1 past the break-point ($\omega = 1/\tau$).

Design, Second Attempt (PD-Control)



Open-loop transfer function: $KD(s)G(s) = \dfrac{K(\tau s + 1)}{s^2}$



For the G-P relationship to be valid, choose the break-point several times smaller than desired ω_c:

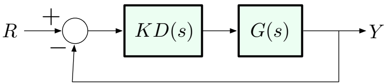
$\Longrightarrow$ let's take $\tau = 10$

$\Longrightarrow \dfrac{1}{\tau} = 0.1 = \dfrac{\omega_c}{5}$

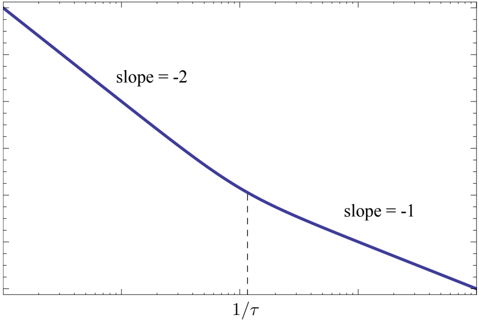
Open-loop t.f.:

$$KD(s)G(s) = \frac{K(10s + 1)}{s^2}$$

Design, Second Attempt (PD-Control)



Open-loop transfer function: $KD(s)G(s) = \dfrac{K(10s+1)}{s^2}$

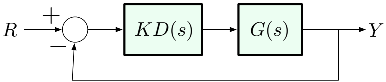


- Want $\omega_c \approx 0.5$
- This means that

$$
\begin{aligned}
M(j0.5) &= 1 \\
|KD(j0.5)G(j.05)| & \\
&= \frac{K|5j+1|}{0.5^2} \\
&= 4K\sqrt{26} \approx 20K
\end{aligned}
$$

$$\Longrightarrow K = \frac{1}{20}$$

PD Control Design: Evaluation



$$G(s) = \frac{1}{s^2}$$

Initial design: $KD(s) = \dfrac{10s+1}{20}$

What have we accomplished?

- PM $\approx 90°$ at $\omega_c = 0.5$
- still need to check in Matlab and iterate if necessary

Trade-offs:

- want ω_{BW} to be large enough for fast response (larger $\omega_{\mathrm{BW}} \longrightarrow$ larger $\omega_n \longrightarrow$ smaller t_r), but not too large to avoid noise amplification at high frequencies
- PD control increases slope $\longrightarrow$ increases $\omega_c \longrightarrow$ increases $\omega_{\mathrm{BW}} \longrightarrow$ faster response
- usual complaint: D-gain is not physically realizable, so let's try lead compensation