ECE 486: Control Systems

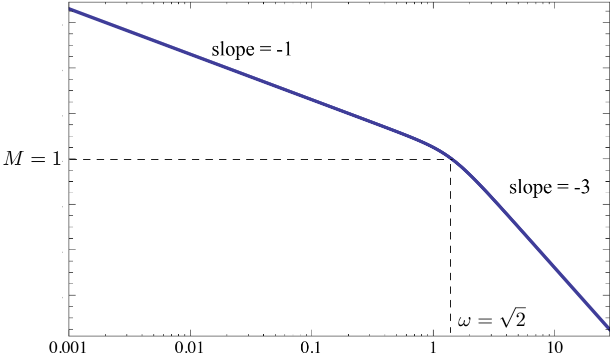
- Lecture 17A: Bode plots for gain/phase margins

Goal: learn to read off gain and phase margins of the closed-loop system from the Bode plot of the open-loop transfer function

Reading: FPE, Section 6.1

Crossover Frequency and Stability

Definition: The frequency at which $M = 1$ is called the *crossover frequency* and denoted by ω_c.

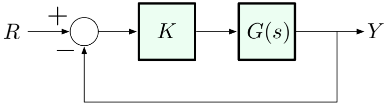


Transition from stability to instability on the Bode plot:

$$\text{for critical } K, \qquad \angle G(j\omega_c) = 180^\circ$$

Stability from Frequency Response

Consider this unity feedback configuration:



Suppose that the *closed-loop* system, with transfer function

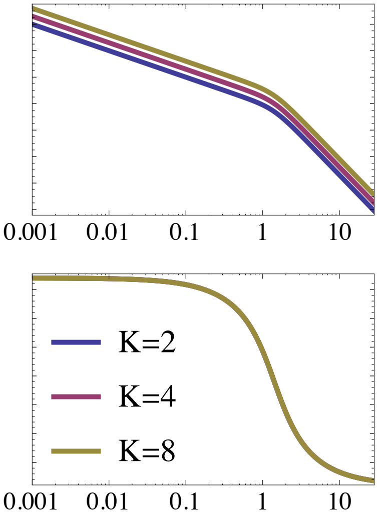
$$\frac{KG(s)}{1 + KG(s)},$$

is stable for a given value of K.

Question: Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).

Effect of Varying K



What happens as we vary K?

- ϕ independent of $K \Longrightarrow$ only the M-plot changes
- If we multiply K by 2:

$$\log(2M) = \log 2 + \log M$$

  – M-plot shifts up by $\log 2$

- If we divide K by 2:

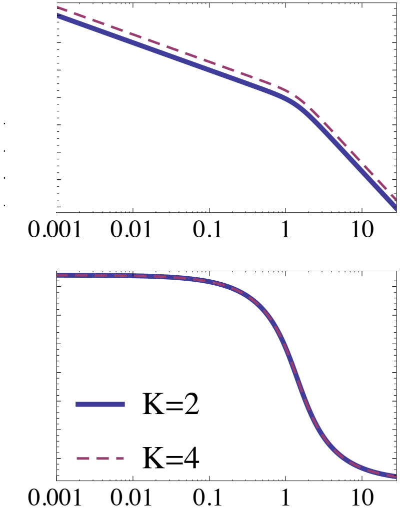
$$\log(\frac{1}{2}M) = \log\frac{1}{2} + \log M = -\log 2 + \log M$$

  – M-plot shifts down by $\log 2$

Changing the value of K moves the crossover frequency ω_c!!

Gain Margin

Back to our example: $G(s) = \frac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)

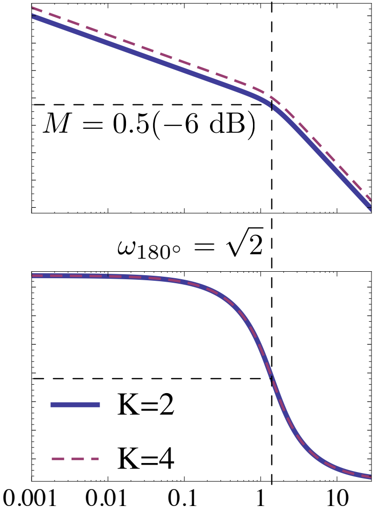


Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180°$

Since varying K doesn't change $\omega_{180°}$, to find GM we need to inspect M at $\omega = \omega_{180°}$

Gain Margin

Our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180^\circ$

Since varying K doesn't change ω_{180°, to find GM we need to inspect M at $\omega = \omega_{180^\circ}$

In this example:

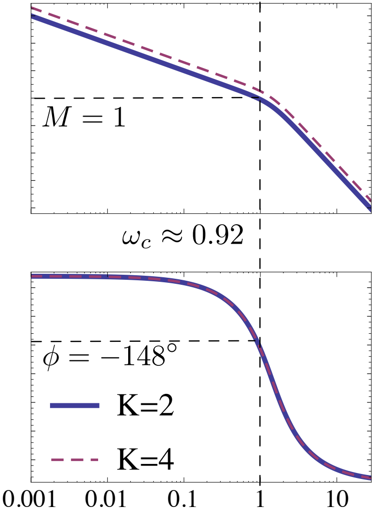
at $\omega_{180^\circ} = \sqrt{2}$

$M = 0.5\,(-6\text{ dB})$,

so GM = 2

Phase Margin

Our example: $G(s) = \dfrac{1}{s(s^2 + 2s + 2)}$, $K = 2$ (stable)



Phase margin (PM) is the amount by which the phase at the crossover frequency ω_c differs from $180° \mod 360°$

To find PM, we need to inspect ϕ at $\omega = \omega_c$

In this example:

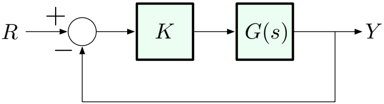
at $\omega_c \approx 0.92$

$$\phi = -148°,$$

so $\text{PM} = (-148°) - (-180°) = 32°$

(in practice, want $\text{PM} \geq 30°$)

Example



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \qquad \zeta, \omega_n > 0$$

Consider gain $K = 1$, which gives closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)} = \frac{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}{1 + \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{— prototype 2nd-order response}$$

Question: what is the gain margin at $K = 1$?

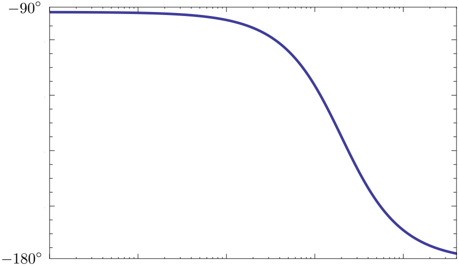
Answer: $\mathrm{GM} = \infty$

Example

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the phase plot:

- starts at $-90°$ (Type 1 term with $n = -1$)
- goes down by $90°$ (Type 2 pole)



Recall: to find GM, we first need to find $\omega_{180°}$, and here there is no such $\omega \implies$ no GM.

Example

So, at $K = 1$, the gain margin of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

is equal to ∞ — what does that mean?

It means that we can keep on increasing K indefinitely without ever encountering instability.

What about phase margin?

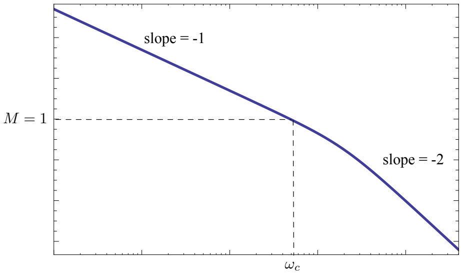
Example: Phase Margin

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the magnitude plot:

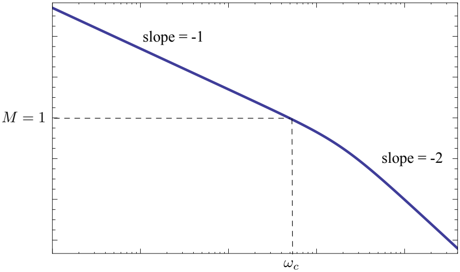
- low-frequency asymptote slope -1 (Type 1 term, $n = -1$)
- slope down by 1 past the breakpt. $\omega = 2\zeta\omega_n$ (Type 2 pole)

$\Longrightarrow$ there is a finite crossover frequency ω_c!!



Example 2: Magnitude Plot

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$



It can be shown that, *for this system*,

$$\text{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)$$

— for PM $< 70°$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin for 2nd-Order System

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right) \approx 100 \cdot \zeta$$

Conclusions:

larger PM $\iff$ better damping

(open-loop quantity) (closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Preview: Bode's Gain-Phase Relationship

In the next lecture, we will see the following more generally:

Hendrik Wade Bode
(1905–1982)

Bode's Gain-Phase Relationship: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a design guideline.