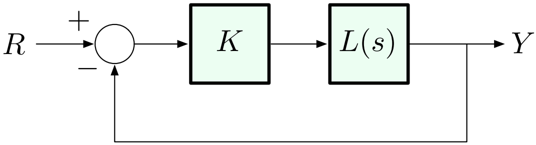
ECE486: Control Systems

- Lecture 12A: Root Locus Rules DEF

Goal: Introduce Root Locus Rules DEF.

Reading: FPE, Chapter 5

Reminder: Root Locus



$$\text{where } L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n},\ m \leq n$$

Root locus: the set of all $s \in \mathbb{C}$ that solve the *characteristic equation*

$$a(s) + Kb(s) = 0$$

as K varies from 0 to ∞.

Equivalent Characterization of RL: Phase Condition

Recall our original definition: The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A point $s \in \mathbb{C}$ is on the RL if and only if

$$L(s) = \underbrace{-\frac{1}{K}}_{\text{negative and real}} \qquad \text{for some } K > 0$$

This gives us an equivalent characterization:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180^\circ$, i.e., $L(s)$ is real and negative.

Reminder: Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

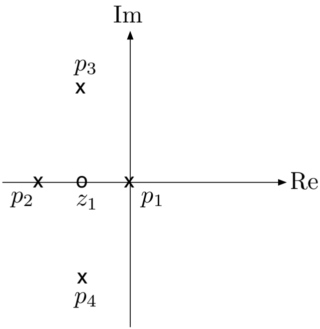
- Rule A — number of branches (= number of open loop poles)
- Rule B — start points (= open loop poles)
- Rule C — end points (= open loop zeros)
- Rule D — real locus (located relative to *real* open-loop poles/zeros)
- Rule E — asymptotes
- Rule F — $j\omega$-crossings

Last time, we have covered Rules A–C

Example

Let's consider $\quad L(s) = \dfrac{s+1}{s(s+2)\big((s+1)^2+1\big)}$

- Rule A: $\begin{cases} m = 1 \\ n = 4 \end{cases} \implies$ 4 branches
- Rule B: branches start at open-loop poles
  $s = 0,\ s = -2,\ s = -1 \pm j$
- Rule C: branches end at open-loop zeros $\qquad s = -1, \pm\infty$



Example, continued

Three more rules:

- Rule D: real locus
- Rule E: asymptotes
- Rule F: $j\omega$-crossings

Rules D and E are both based on the fact that

$$1 + KL(s) = 0 \text{ for some } K > 0 \quad \Longleftrightarrow \quad L(s) < 0$$

Characteristic equation in our example:

$$\underbrace{s(s+2)\big((s+1)^2+1\big)}_{a(s)} + K \underbrace{(s+1)}_{b(s)} = 0$$

$$s^4 + 4s^3 + 6s^2 + (4+K)s + K = 0$$

— don't even think about factoring this polynomial!!

Rule D: Real Locus

The branches of the RL start at the open-loop poles. Which way do they go, left or right?

Recall the phase condition:

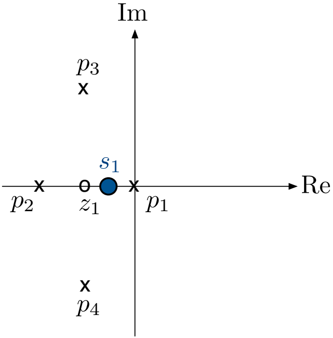
$$1 + KL(s) = 0 \qquad \Longleftrightarrow \qquad \angle L(s) = 180^\circ$$

$$\begin{aligned}
\angle L(s) &= \angle \frac{b(s)}{a(s)} \\
&= \angle \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} \\
&= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j)
\end{aligned}$$

— this sum must be $\pm 180^\circ$ for *any* s that lies on the RL.

Rule D: Real Locus

So, we try test points:



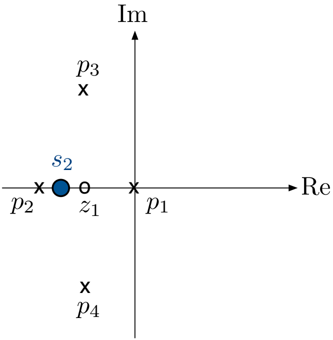
$$\angle(s_1 - z_1) = 0^\circ \quad (s_1 > z_1)$$
$$\angle(s_1 - p_1) = 180^\circ \quad (s_1 < p_1)$$
$$\angle(s_1 - p_2) = 0^\circ \quad (s_1 > p_2)$$
$$\angle(s_1 - p_3) = -\angle(s_1 - p_4)$$

(conjugate poles cancel)

$$\angle(s_1 - z_1) - [\angle(s_1 - p_1) + \angle(s_1 - p_2) + \angle(s_1 - p_3) + \angle(s_1 - p_4)]$$
$$= 0^\circ - [180^\circ + 0^\circ + 0^\circ] = -180^\circ \qquad \checkmark s_1 \text{ is on RL}$$

Rule D: Real Locus

Try more test points:



$$\angle(s_2 - z_1) = 180^\circ \quad (s_2 < z_2)$$
$$\angle(s_2 - p_1) = 180^\circ \quad (s_2 < p_1)$$
$$\angle(s_2 - p_2) = 0^\circ \quad (s_2 > p_2)$$
$$\angle(s_2 - p_3) = -\angle(s_1 - p_4)$$

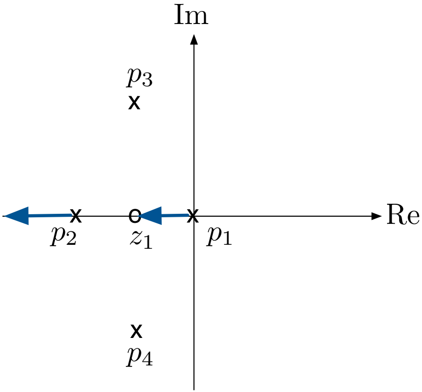
(conjugate poles cancel)

$$\angle(s_2 - z_1) - [\angle(s_2 - p_1) + \angle(s_2 - p_2) + \angle(s_2 - p_3) + \angle(s_2 - p_4)]$$
$$= 180^\circ - [180^\circ + 0^\circ + 0^\circ] = 0^\circ$$

✗ s_2 is not on RL

Rule D: Real Locus

Rule D: If s is *real*, then it is on the RL of $1 + KL$ if and only if there are an odd number of *real open-loop poles and zeros* to the right of s.



Rule E: Asymptotes

How does the locus look as $s \to \infty$?

$$\begin{aligned} 180^\circ = \angle L(s) &= \angle \frac{s^m + b_1 s^{m-1} + \dots}{s^n + a_1 s^{n-1} + \dots} \\ &= \angle \frac{s^{m-n} + b_1 s^{m-n-1} + \dots}{1 + a_1 s^{-1} + \dots} \\ &\simeq \angle s^{m-n} \ \text{ if } |s| \to \infty \qquad (\text{recall } m \le n) \end{aligned}$$

Claim: If $\angle s^{m-n} = 180^\circ$, then

$$\angle s = \frac{180^\circ + \ell \cdot 360^\circ}{n-m}, \qquad \ell = 0, 1, \dots, n-m-1$$

Proof:

$$s = |s| e^{j\angle s} \qquad s^{m-n} = |s|^{m-n} e^{j(m-n)\angle s}$$

$$\angle s^{m-n} = 180^\circ \qquad \Longrightarrow \qquad (m-n)\angle s = 180^\circ + \ell \cdot 360^\circ$$

Rule E: Asymptotes

Rule E: Branches near ∞ have phase

$$\angle s \simeq \frac{180^\circ + \ell \cdot 360^\circ}{n-m}$$
$$= \frac{(2\ell+1)\cdot 180^\circ}{n-m}, \qquad \ell = 0, 1, \dots, n-m-1$$

Note: if $m = n$, then there are no branches at ∞.

Back to Example: Rule E

Branches near ∞ have phase

$$\angle s = \frac{(2\ell+1)\cdot 180^\circ}{n-m}, \qquad \ell = 0, 1, \dots, n-m-1$$

In our example, $L(s) = \dfrac{s+1}{s(s+2)\big((s+1)^2+1\big)}$ $\qquad \begin{cases} n = 4 \\ m = 1 \end{cases}$

$$\angle s = \frac{(2\ell+1)\cdot 180^\circ}{3}, \qquad \ell = 0, 1, 2$$

$$\ell = 0: \qquad \frac{2\cdot 0+1}{3}180^\circ = 60^\circ$$

$$\ell = 1: \qquad \frac{2\cdot 1+1}{3}180^\circ = 180^\circ$$

$$\ell = 2: \qquad \frac{2\cdot 2+1}{3}180^\circ = \frac{5}{3}180^\circ = \left(2 - \frac{1}{3}\right)180^\circ = -60^\circ$$

— asymptotes have angles 60°, 180°, -60°

Rule F: $j\omega$-crossings

Do the branches of the root locus cross the $j\omega$ axis?
(transition from *stability* to *instability*)

Goal: determine if the equation

$$a(j\omega) + Kb(j\omega) = 0$$

has a solution $\omega \geq 0$ for some $K > 0$.

Best approach here: use the *Routh test* to first determine the critical value of K (when the characteristic polynomial becomes unstable), then plug it in and solve for $j\omega$-crossings (numerically or analytically).

Rule F: $j\omega$-crossings

In our example, the characteristic polynomial is

$$s^4 + 4s^3 + 6s^2 + (4 + K)s + K$$

Form the Routh array:

$$\begin{array}{lccc} s^4 : & 1 & 6 & K \\ s^3 : & 4 & 4+K & 0 \\ s^2 : & 20-K & 4K & \\ s^1 : & 80-K^2 & 0 & \\ s^0 : & 4K & & \end{array}$$

For stability, need $20 - K > 0$, $80 - K^2 > 0$, $4K > 0$

The characteristic polynomial is stable for $K < \sqrt{80} = 4\sqrt{5}$

$$\Longrightarrow \ K_{\text{critical}} = 4\sqrt{5}$$

Rule F: $j\omega$-crossings

In our example, the characteristic polynomial is

$$s^4 + 4s^3 + 6s^2 + (4 + K)s + K$$

The critical value: $K = 4\sqrt{5}$ (from Routh test).

To find the $j\omega$-crossing, plug in and solve:

$$(j\omega)^4 + 4(j\omega)^3 + 6(j\omega)^2 + (4 + 4\sqrt{5})j\omega + 4\sqrt{5} = 0$$

$$\omega^4 - 4j\omega^3 - 6\omega^2 + (4 + 4\sqrt{5})j\omega + 4\sqrt{5} = 0$$

real part: $\omega^4 - 6\omega^2 + 4\sqrt{5} = 0$

imag. part: $-4\omega^3 + 4(1 + \sqrt{5})\omega = 0 \qquad \omega^2 = 1 + \sqrt{5}$

$j\omega$-crossing at $j\omega_0 = \sqrt{1 + \sqrt{5}} \approx 1.8$, when $K = 4\sqrt{5} \approx 8.9$

Complete Root Locus

$$L(s) = \frac{s+1}{s(s+2)\big((s+1)^2+1\big)}$$

Rule A: 4 branches

Rule B: branches start at $p_1, \ldots, p_4$

Rule C: branches end at $z_1, \pm\infty$

Rule D: real locus $= [z_1, p_1] \cup (-\infty, p_2]$

Rule E: asymptotes form angles at $60°, 180°, -60°$

Rule F: $j\omega$-crossings at $\pm j\omega_0$, where

$$\omega_0 = \sqrt{1+\sqrt{5}} \approx 1.8$$

when $K = 4\sqrt{5} \approx 8.9$

(transition from stability to instability)