Problem 1. Consider the single-input, single-output transfer function:

$$Y(s) = \frac{s+1}{s^2+2s+2}U(s)$$

(a) Find a second-order state-space model that represents this transfer function.
(b) For this state-space model, calculate a state-feedback controller $u = -Kx + r$ that places the closed-loop poles at -4 and -25.
(c) Construct a stable observer to estimate x based on the known inputs u and observations y. You may use MATLAB for this part.
(d) With the controller and observer from the previous problems in place, calculate k_r such that $u = -K\hat{x} + k_r r$ yields a closed-loop system Y/R with unity gain. You may use MATLAB.
(e) Plot the step response using MATLAB.

Solution.

(a) Recall that for a transfer function:

$$\frac{Q(s)}{P(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

the controllable canonical realization is:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T \tag{2}$$

$$C = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1}b_0 & b_{n-2} - a_{n-2}b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \tag{3}$$

$$D = [b_0] \tag{4}$$

Here $b_0 = 0$ and therefore we have that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(b) If we want to place the closed loop poles at -4 and -25, then the new denominator would be $(s+4)(s+25) = s^2 + 29s + 100$. Therefore, $a_0 + k_1 = 100 \implies k_1 = 100 - 2 = 98$ and $a_1 + k_2 = 29 \implies k_2 = 29 - 2 = 27$. Hence, $K = \begin{bmatrix} 98 & 27 \end{bmatrix} \implies u = -\begin{bmatrix} 98 & 27 \end{bmatrix} x + r$.

(c) For an observer gain L, the observer poles are the eigenvalues of $A - LC$, which coincide with the eigenvalues of $A^T - C^T L^T$. Consequently to compute the observer gain in MATLAB, we apply the `place` command for (A^T, C^T). It is a rule of thumb, to pick observer poles to be 2-5 times further than the controller poles. Suppose we want to place the observer poles at $\{-50, -51\}$. By using the MATLAB command

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L = place(A', C', [-50, -51])
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where (A^T, C^T) are as above we get $L = \begin{bmatrix} -2449 & 2548 \end{bmatrix}^T$.

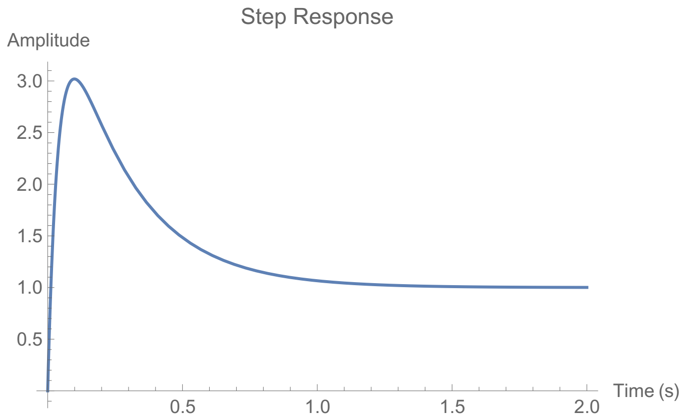
(d) To obtain k_r, recall that $\hat{x}(t) \equiv 0$ if $\hat{x}(0) = 0$, so we can ignore the observer. The closed loop system transfer function disregards initial conditions. With full state feedback, the closed loop transfer function is

$$\frac{Y(s)}{R(s)} = C\left[sI - (A - BK)\right]^{-1} Bk_r$$

To set the DC gain to unity we need

$$\begin{aligned} 1 &= C\left[sI - (A - BK)\right]^{-1} Bk_r \\ \implies k_r &= 100 \end{aligned}$$

(e) The plot is shown below. The overshoot is expected due to the LHP zero.



Problem 2. Consider the single-input, single-output transfer function:

$$G_p(s) = \frac{1 - s/2}{1 + s/2}\frac{1}{s^2}$$

(a) Find a third-order state-space model that represents this transfer function.
(b) For this state-space model, calculate a state-feedback controller $u = -Kx + r$ that places the closed-loop poles at -4, -13, and -25. You may use MATLAB to calculate this controller, but not to find the the state-space model.
(c) Construct a stable observer, and put this together to form a compensator of the form $U = -G_c Y + G_r R$. You may use MATLAB.
(d) Calculate the Nyquist plot of $G_c G_p$. You may use MATLAB to do so. Is the system stable? If so, calculate the gain and phase margins.

Solution. (a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$

(b) $K = \begin{bmatrix} 1300 & 477 & 40 \end{bmatrix}$

(c) We place the observer poles at -50, -51, -52. $L = \begin{bmatrix} 9301 \\ 18450 \\ 29400 \end{bmatrix}$.

$$\begin{aligned} G_c &= \frac{2.207e07s^2 + 7.979e07s + 8.619e07}{s^3 + 193s^2 + 1.432e04s + 2.257e07} \\ G_r &= \frac{s^3 + 153s^2 + 7802s + 1.326e05}{s^3 + 193s^2 + 1.432e04s + 2.257e07}. \end{aligned}$$

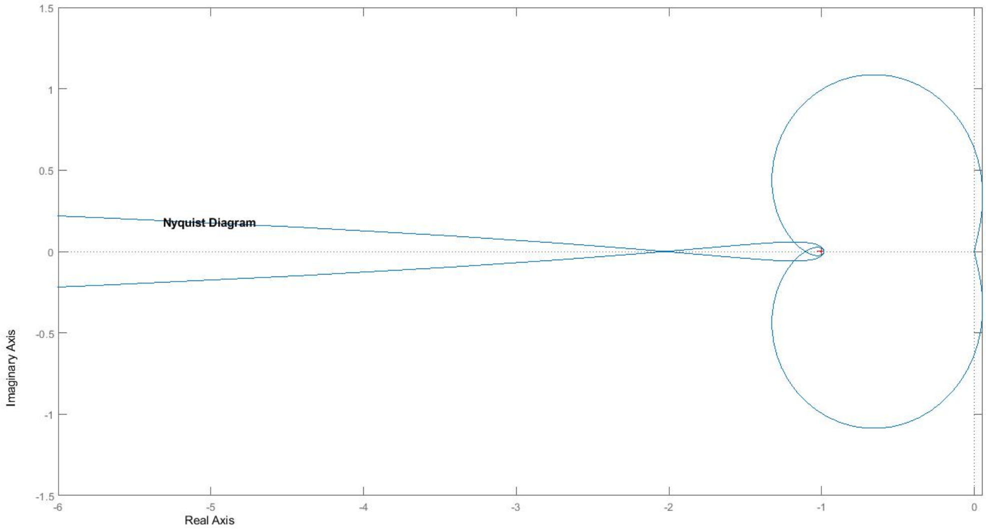


Figure 1: Nyquist plot in Q2

(d) See Figure 1 for the Nyquist plot of G_cG_p. The gain margin is 0.115 dB and the phase margin is -1.11 deg.
We now check stability with the nyquist plot. I have redrawn the nyquist plot in Figure 2 for counting the number of windings around -1. Clearly there are 3 anticlockwise loops around -1, coloured in red, green and blue. So N must be -3. G_c has 2 RHP poles, and G_p has 2 poles at zero. If the poles at zero are included as RHP poles, $P = 4$, else $P = 2$. In either case, $Z = N + P = 1/-1$ and not zero. However, the closed loop has to be stable because we placed the poles in the LHP ourselves. ¡

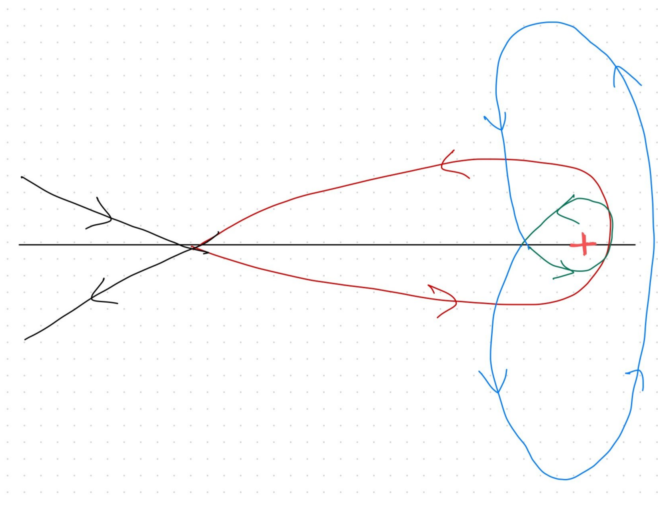
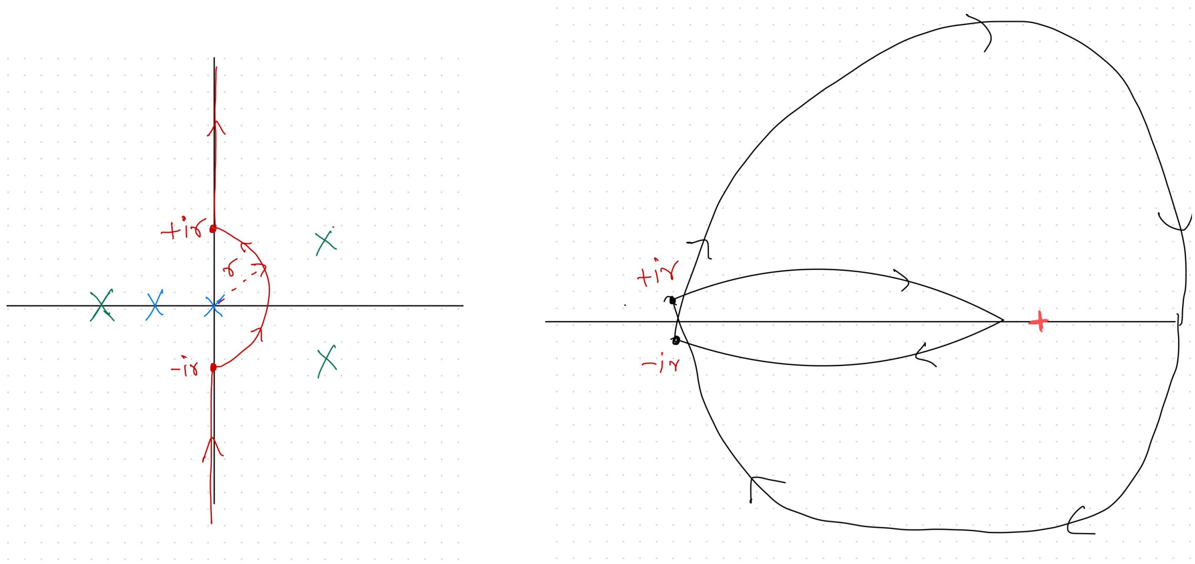


Figure 2: Rough Nyquist plot for counting N

The problem here is that G_c has a double pole at the origin. This is why the nyquist plot is not a closed loop, and shoots to infinity as ω approaches zero. So we need to consider a small semi-circular modified contour of radius r around the origin as shown in Figure 3a. Radius r is taken to be very small. The green crosses in Figure 3a represent the poles of G_c, two of which are in RHP and one is in LHP. The blue crosses represent poles of G_p, one in LHP and two at the origin. However, with respect to the modified contour, the two poles at the origin lie in the LHP. So they are not counted as open loop RHP poles and $P = 2$.

Now we need to understand how the image of the Nyquist plot changes under this modified contour. See Figure 3b to see what happens. We forget the 3 anticlockwise loops already accounted for before and focus only on the black part that shoots off to infinity in Figure 2. For r very small, $G_pG_c(jr) \approx k.\frac{1}{(jr)^2} = \frac{-k}{r^2}$ where k is the finite limit $\lim_{s\to 0} s^2 G_cG_p(s)$. The point is that $G_cG_p(jr)$ will be close to the real line even though its magnitude is very large $\approx \frac{k}{r^2}$. Since the plot goes toward negative infinity, we can also say that k is negative. The semicircular contour can be parametrized as $re^{j\theta}$ as θ goes from $-\pi/2 \to -\pi/4 \to 0 \to \pi/4 \to \pi/2$. Since r is really small, $G_cG_p(re^{j\theta}) \approx \frac{-k}{r^2}e^{-2j\theta}$. Since $-k$ is positive, the phase of $G_cG_p(re^{j\theta})$ will be $\approx e^{-2j\theta}$. As θ travels from $-\pi/2$ to $\pi/2$ in the anticlock direction, -2θ travel from $-\pi$ back to $\pi = -\pi$ in the clockwise direction. So the nyquist plot travels through a large clockwise circle of radius $\frac{-k}{r^2}$ as indicated in Figure 3b when we modify the contour. This circle will wind around -1 due to its very large radius, which makes $N = -3 + 1 = -2$. Since $P = 2$, $Z = N + P = 0$ which means the system is stable.



(a) Modified contour to avoid poles at origin (b) Image of the modified contour

Figure 3: Modified countour and its image.

Problem 3. Consider the following nonlinear system

$$\dot{x} = -4x - 2u + u^3$$

(a) Find an equilibrium $(\bar{x}, \bar{u})$ with $\bar{u} = 2$.

(b) Linearize the dynamics around $(\bar{x}, \bar{u})$.

Solution. (a) For equilibrium, we should have

$$-4\bar{x} - 2\bar{u} + \bar{u}^3 = 0.$$

Substituting $\bar{u} = 2$ gives $\bar{x} = 1$. Hence the equilibrium point with $\bar{u} = 2$ is $(\bar{x}, \bar{u}) = (1, 2)$.

(b) First, let us define deviations from the equilibrium point:

$$\delta_x := x - \bar{x}, \qquad \delta_u := u - \bar{u}.$$

For the above system, we have $f(x,u) = -4x - 2u + u^3$. From (a) we know that $(\bar{x}, \bar{u}) = (1, 2)$. Hence we have :

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = -4, \tag{5}$$

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = -2 + 3 \times 2^2 = 10. \tag{6}$$

Therefore, we can obtain the following linear system via linearization:

$$\dot{\delta}_x = -4\delta_x + 10\delta_u.$$