Problem 1 (20 points). Calculate the transfer function for the following state-space model.

$$\dot{x} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} u$$

Solution.
Recall that for a system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{1}$$

the transfer function is given by

$$H(s) = C(sI - A)^{-1} B + D \tag{2}$$

Here, $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $D = [3]$.
Therefore,

$$H(s) = \frac{3s^2 - 19s + 14}{s^2 - 6s + 1}$$

Problem 2 (20 points). For the following transfer function, calculate the controllable canonical form (CCF) state-space model.

$$G(s) = \frac{1}{(s+2)(s^2 + 2s + 5)}$$

Solution.
The denominator can be expanded as $s^3 + 4s^2 + 9s + 10$. Recall that for a transfer function:

$$\frac{Q(s)}{P(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

the controllable canonical realization is:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \tag{3}$$

$$B = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T \tag{4}$$

$$C = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & b_{n-2} - a_{n-2} b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \tag{5}$$

$$D = [b_0] \tag{6}$$

Therefore we have that,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -9 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + [0]u$$

Problem 3 (25 points). Prove the following:

(a) $(AB)^\top = B^\top A^\top$

(b) $(ABC)^\top = C^\top B^\top A^\top$

(c) $(A^\top)^{-1} = (A^{-1})^\top$

(d) $(I - TAT^{-1})^{-1} = T(I - A)^{-1}T^{-1}$

(e) For any integer $k \geq 0$, $(TAT^{-1})^k = TA^kT^{-1}$

Here, $(\cdot)^\top$ denotes the transpose operator and $(\cdot)^{-1}$ denotes the inverse operator. Assume matrices are invertible whenever needed.

Recall that the definition of A^{-1} is the unique matrix such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

You may use previous parts to prove later parts, e.g. you may invoke Part (a) when proving Part (b).

Solution.
Note: To prove something means to definitely establish a result for *all* cases. Therefore, in these questions, constructing a few $m_i \times n_i$ where $m_i, n_i \in \{1, 2, 3, 4, \dots\}$ example matrices and showing the identity holds is **not** a proof. In the following $(\cdot)^\top = (\cdot)^T$

(a) Let M_{ij} denote the element in row i and column j of a matrix M. Then note that $M_{ji} = (M^T)_{ij}$. Therefore,

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = \left(B^T A^T\right)_{ij}$$

(b) By (a) we have,

$$(ABC)^T = (A\,(BC))^T = (BC)^T\,A^T = C^T B^T A^T$$

(c) If $B := A^{-1}$ is the inverse of A then $AB = I = BA$. Since the identity matrix is symmetric, in particular, taking the transpose by (a) we have,

$$B^T A^T = (AB)^T = I = (BA)^T = A^T B^T$$

and $B^T = (A^T)^{-1}$.

(d) Since we want to establish that

$$(I - TAT^{-1})^{-1} = T(I - A)^{-1}T^{-1}$$

multiply the RHS with $\left(I - TAT^{-1}\right)$. We get,

$$\begin{aligned}
\left(I - TAT^{-1}\right)\left(T(I - A)^{-1}T^{-1}\right) &= \left(TIT^{-1} - TAT^{-1}\right)\left(T\,(I - A)^{-1}\,T^{-1}\right) \\
&= \left(T\,(I - A)\,T^{-1}\right)\left(T\,(I - A)^{-1}\,T^{-1}\right) \\
&= T\,(I - A)\,(I - A)^{-1}\,T^{-1} \\
&= TT^{-1} \\
&= I
\end{aligned}$$

(e) The statement trivially holds for $k = 0$. Assume now that it holds for some $k = n > 0$. Then,

$$\left(TAT^{-1}\right)^{k+1} = \left(TAT^{-1}\right)^k\left(TAT^{-1}\right) = \left(TA^kT^{-1}\right)\left(TAT^{-1}\right) = TA^{k+1}T^{-1}$$

Therefore, by induction it holds true for all $k \in \mathbb{N}$.

Problem 4 (30 points)**.** Determine whether or not the following systems are controllable. If they are controllable, put them in controllable canonical form.

(a)

$$\dot{x} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 5 & 1 & 3 & 2 \\ 6 & 1 & 3 & 4 \\ 1 & 7 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u$$

(b)

$$\dot{x} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Solution. (a) This system is not controllable (B matrix is the zero matrix).

(b) This system is controllable since the controllability matrix reads as $\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$ which is full rank. From (1) and (2) the transfer function is given by $\dfrac{s+1}{s^2-6s+1}$. Then from (3) - (6) we have that the controllable canonical form is:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + [0] u$$