

Problem 1 (20 points). Suppose we have designed a controller K for a plant G . After simulations, we think the response speed for a unit step reference signal is too slow. In this case, shall we decrease or increase the crossover frequency of the open loop transfer function? What issues will there be if we have a very large crossover frequency?

Solution.

To make the response speed faster, we should increase the crossover frequency of the open loop transfer function to have a larger bandwidth. If we have a very large crossover frequency, we may end up with a poor noise rejection and small phase margin, which may cause the instability of the system.

Problem 2 (80 points). DC motors are used in many electro-mechanical systems. A model for a DC motor is:

$$J\dot{y} + by = cV \quad (1)$$

where y is the angular velocity of the motor shaft (deg/sec), and V is the input voltage (Volts). The model parameters are: J =Rotational Inertia (N m sec²/deg²), b =rotational damping (Nm sec/deg), and c =gain from input voltage to applied torque (Nm/Volts). For this problem we'll use the values $J = 3$, $b = 5$ and $c = 12$.

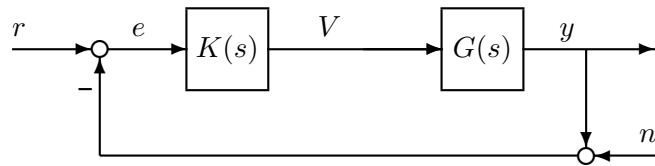


Figure 1: DC Motor Feedback System

Consider the feedback loop in the figure above where $G(s)$ is the transfer function for the DC motor and $K(s)$ is the controller transfer function. The specifications are to design a controller so that the closed loop:

- i. is stable,
- ii. has a loop crossover frequency near 20 rad/sec
- iii. has gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec.
- iv. has gain less than 0.04 from input n to output y for frequencies above 200 rad/sec.

Perform the following steps:

- (a) Translate specifications iii. and iv. into requirements on the loop gain $|L(j\omega)|$.

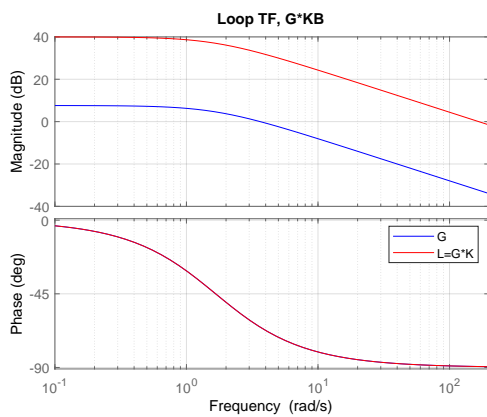
Answer: The transfer function from r to e is $S(s) = \frac{1}{1 + L(s)}$. The steady error due to a unit step reference input $r(t) = \bar{r}$ is given by $\bar{e} = S(0)\bar{r}$. Thus specification iii) is equivalent to $|S(j\omega)| \leq 0.01$ for $\omega \leq 0.1 \frac{\text{rad}}{\text{sec}}$. This is approximately equivalent to $|L(j\omega)| \geq 100$ for $\omega \leq 0.1 \frac{\text{rad}}{\text{sec}}$.

The transfer function from n to y is $-T(s) = \frac{-L(s)}{1 + L(s)}$. Specification iv. corresponds to $|T(j\omega)| \leq 0.04$ for $\omega \geq 200 \frac{\text{rad}}{\text{sec}}$. and this is approximately equivalent to $|L(j\omega)| \leq 0.04$ for $\omega \geq 200 \frac{\text{rad}}{\text{sec}}$.

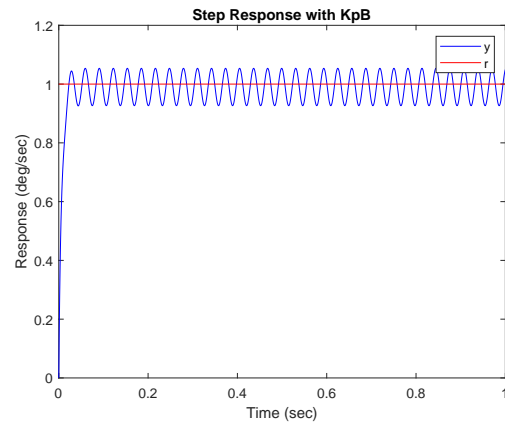
- (b) Design a proportional control law $K(s) = K_p$ to satisfy requirements i. and iii. only. Plot $G(s)$ and $L(s) = G(s)K(s)$ on the same Bode plot using the Matlab command `bode`. Also, simulate the closed-loop system with $r(t) = 1$ deg/sec and $n(t) = 0.1 \sin(200t)$. Plot the motor speed response $y(t)$ and the reference command $r(t)$ on the same plot. Hand in both your Bode plot and your step response plot. For your choice of K_p , what is the loop cross-over frequency and what is the value of $|L(j200)|$?

Note: An m-file and Simulink diagram have been posted with this homework. You only need to enter your proportional gain and the m-file will generate both the Bode plot and the step response plot.

Answer: In the previous part we showed that $|L(j\omega)| = |G(j\omega)K(j\omega)| \geq 100$ for $\omega \leq 0.1 \frac{\text{rad}}{\text{sec}}$ will ensure specification iii. is satisfied. The system transfer function is $G(s) = \frac{12 \text{ sec}}{3s + 5}$. This has a corner frequency at $\omega = 1.6 \frac{\text{rad}}{\text{sec}}$ and DC gain $G(0) = \frac{12}{5} = 2.4$. Thus $K_p = \frac{100}{2.4}$ will ensure that iii. is approximately satisfied. You can verify that the closed-loop is stable for this gain and hence i. is also satisfied. The Bode plot and step responses are shown below. They were generated using the m-file distributed with this homework. From the Bode plot the cross-over frequency is approximately 165 rad/sec . It also clear from the Bode plot that the loop gain at 200 rad/sec is far above the requirement $|L(j200)| \leq 0.04$ that we derived from specification iv. Thus this design fails the noise filtering specification and hence the time response shows a large amplitude sinusoid due to the noise.



(a) Part B Bode Plot

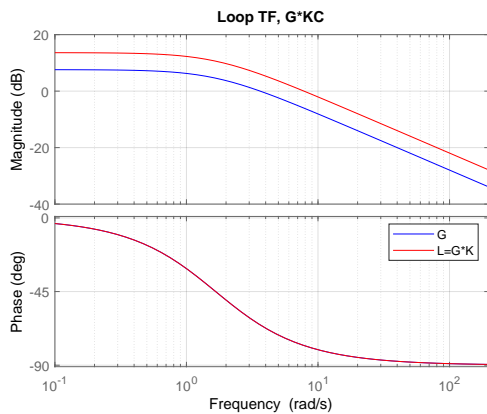


(b) Part B Step Response

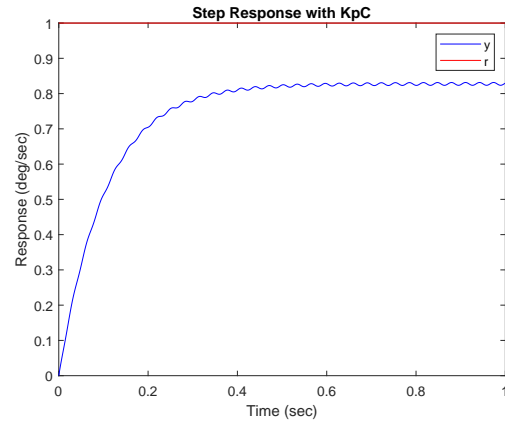
- (c) Next, design a proportional control law $K(s) = K_p$ to satisfy requirements i. and iv. only. Generate and hand-in the Bode plot and step response plots as described in the part (b). For your choice of K_p , what is the loop cross-over frequency and what is the value of $|L(j0)|$?

Answer: In part (a) we showed that $|L(j\omega)| = |G(j\omega)K(j\omega)| \leq 0.04$ for $\omega \geq 200 \text{ rad/sec}$ will ensure specification iv. is satisfied. Thus $K_p \leq \frac{0.0385}{|G(j200)|}$ will ensure that iv. is satisfied. You can verify that the closed-loop is stable for this gain and hence i. is also satisfied. The Bode plot and step responses are shown below. From the Bode plot the cross-over frequency is approximately 7.5 rad/sec . It also clear from the Bode plot that the loop gain at 0 rad/sec is below the requirement $|L(0)| \geq 100$ that we derived from specification iii. Thus this design fails the specification iii. and hence the time response

shows a large steady state error.



(a) Part C Bode Plot



(b) Part C Step Response

- (d) At this point it should be clear that proportional control will not be able to satisfy the design objectives. Use the loop-shaping procedure described in class to design a controller that satisfies objectives ii., iii., and iv. Verify that the closed-loop is stable with your final control design. Generate and hand-in the Bode plot and step response plots as described in the part (b). Your controller will be specified as a transfer function and you'll need to modify the m-file and Simulink diagram.

Recommendation: Use a proportional gain to set the loop cross-over frequency (requirement ii.). Then use an integral boost and roll-off to satisfy requirements iii. and iv., respectively. This may require some iteration to get values that meet all design specifications.

Answer: We can follow the procedure described in the recommendation above. First choose the proportional gain so that the loop gain has crossover $|L(j\omega_c)| = 1$ at $\omega_c = 20$ rad/sec. This can be done with the gain $K_1 = \frac{1}{|G(j\omega_c)|} \approx 5.02$. This will give the first loopshape $L_1(s) = G(s)K_1$.

The first loop shape has DC gain $|L_1(0)| = 12.0$. Thus we need to increase the gain by a factor of $\frac{100}{12.0} = 8.4$ to ensure that we satisfy requirement iii. We'll use an integral boost:

$$K_2(s) = \frac{s + \bar{\omega}_2}{s} \quad (2)$$

where $\bar{\omega}_2$ is the frequency below which the gain starts to increase. With some iteration the choice $\bar{\omega}_2 = 5$ rad/sec seem to easily exceed the low frequency requirement. Notice that $\bar{\omega}_2 < \omega_c$. As discussed in class, for stability and robustness reasons we need the slope of $|L|$ to be "shallow" near ω_c . Thus we need to choose our low-frequency boost to start sufficiently below ω_c that it doesn't have a significant impact on the slope at ω_c . After this stage our controller is $K_1K_2(s)$ and our loop-shape is $L_2(s) = G(s)K_1K_2(s)$.

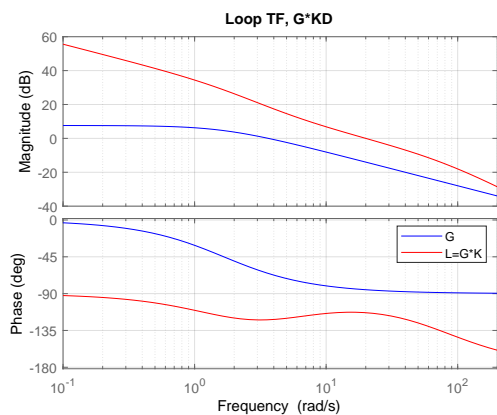
Finally, we need to modify the loop shape to satisfy the design requirement iv. We can choose a roll-off to decrease the high frequency gain.

$$K_3(s) = \frac{\bar{\omega}_3}{s + \bar{\omega}_3} \quad (3)$$

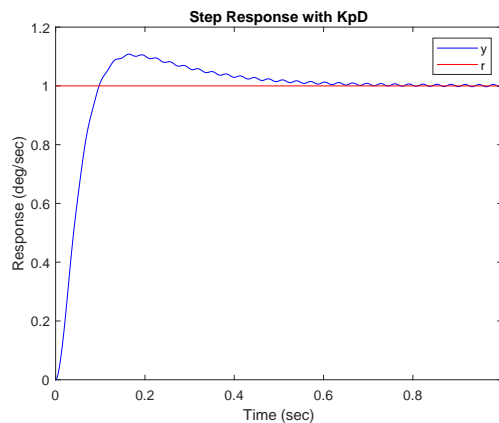
Again, we want to choose $\bar{\omega}_3 > \omega_c$ so that the roll-off has negligible effect on the slope of $|L|$ near cross-over. However, we also want to choose $\bar{\omega}_3$ small enough that we decrease the high frequency gain enough to satisfy $|L(j200)| \leq 0.0385$. After some trial and

error $\bar{\omega}_3 = 80$ rad/sec ensures that the requirements are satisfied. Our final controller is $K(s) = K_1 K_2(s) K_3(s)$ and our final loop-shape is $L_2(s) = G(s)K(s)$.

The Bode plot and step responses are shown below. The step response has a small steady-state error and good noise rejection. The response also has a small overshoot.



(a) Part D Bode Plot



(b) Part D Step Response

- (e) What is the ordinary differential equation that models the input-output dynamics of the control law designed in part (d)?

Answer: The transfer function for the final controller designed in part (d) is:

$$K(s) = 5.02 \frac{s + 5}{s} \frac{80}{s + 80} \quad (4)$$

Multiplying this out using Matlab:

$$K(s) = \frac{401.4s + 2007}{s^2 + 80s} \quad (5)$$

This is a second order transfer function. This transfer function corresponds to the following ODE that relates the controller input e to output u :

$$\ddot{u} + 80\dot{u} = 401.4\dot{e} + 2007e \quad (6)$$