Problem 1. (25 points) Suppose we have the open-loop transfer function $G(s) = \dfrac{K}{s(s+20)(s+85)}$, and we put it through unity feedback, i.e. the closed loop transfer function is $\dfrac{G(s)}{1+G(s)}$.

(a) Set the gain K so that the magnitude is 1 (0 dB) at $\omega = 1$. What value of K achieves this?
(b) We wish to achieve 15% overshoot in the transient response for a step input. What phase margin is required to achieve this?
(c) What frequency on the Bode phase diagram yields this phase margin?
(d) Find the adjusted gain necessary to produce the required phase margin.
(e) Using MATLAB, plot the step response of the compensated system. Did you meet the design specs?

Solution. (a) For 0 db at $\omega = 1$, we have that,

$$|G(j1)| = \left|\frac{K}{j(j+20)(j+85)}\right| = 1$$

so, $K = |j(j+20)(j+85)| = \sqrt{2897626} = 1702.24$

(b) For 15% overshoot, we have that,

$$\zeta = \frac{-\ln(0.15)}{\sqrt{\pi^2 + \ln^2(0.15)}} = 0.5169$$

using this damping ratio, we can obtain the desired phase margin

$$\Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right) = 53.17°$$

(c) The phase which yields this phase margin is $\phi = -180 + \Phi_M = -127$. Dragging the cursor on the Bode plot we find the phase is -127 when $\omega = 11.2$ rad/s. See Figure 1 for details.

(d) We achieve the desired phase margin by shifting the magnitude at $\omega_{\text{PM}} = 11.2$ to 0 dB. This can be done by changing the gain:

$$0\,\text{dB} = 20\log|K^* G(j\omega_{\text{PM}})|$$
$$1 = |K^* G(j\omega_{\text{PM}})|$$

Therefore $K^* = 1/|G(j\omega_{\text{PM}})| = 12.93$ and the adjusted gain is $K^* = \sqrt{2897626} \times 12.93 \approxeq 22010$.

(e) The overshoot is 14.5 % so we have met the design specifications. See Figure 1.

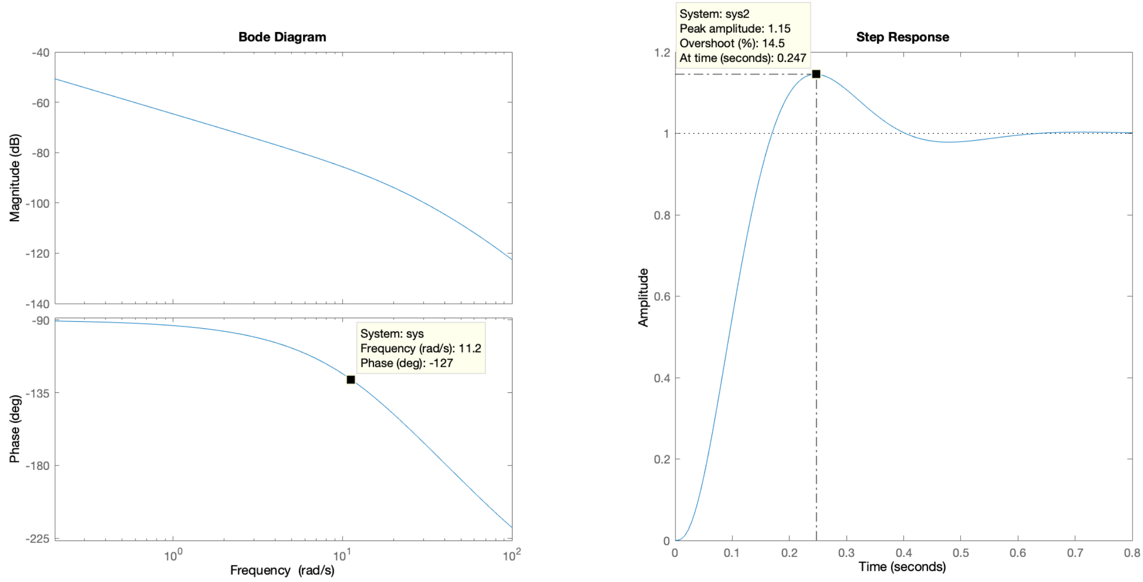

Figure 1: **Left:** Bode-plot and required phase margin and **Right:** Step response of final system

Problem 2. (25 points) For a transfer function $G(s) = \dfrac{K}{s(s+20)(s+85)}$ in unity feedback, design a lag compensator to reduce the steady-state error for a ramp input by a factor of 10 while maintaining a 15% overshoot in the transient response for a step input.

(a) Determine the steady state error of the gain-adjusted system designed in the previous problem.
(b) Find a gain K to satisfy the steady-state specification and plot the Bode plot in MATLAB for this gain.
(c) Determine the phase margin to achieve the desired overshoot. Find the frequency where the phase margin is 10 degrees greater than this phase margin.
(d) Design a lag compensator to achieve a gain of 1 (0 dB) at this frequency. In particular, set a high-frequency asymptote so that the compensated system will have a gain of 1 at this frequency, set the low-frequency asymptote to be at 1 (0 dB), and then connect the two with a -1/decade line (-20 dB/decade line). Find the lag compensator that achieves this.
(e) Why do we increase the phase margin above the desired margin when designing a lag compensator? Did you meet the design specifications? Include relevant plots.

Solution. (a) We have that

$$e_{\text{ss}}^{\text{ramp}} = \frac{1}{\lim\limits_{s\to 0} G(s)} = \frac{(20)(85)}{22010} = 0.0772$$

(b) We want $e_{\text{ss}}^{\text{ramp}} = 0.00772$. But $e_{\text{ss}}^{\text{ramp}} = \dfrac{(20)(85)}{K}$, so $K = 220100$

(c) The desired phase margin depends only on the desired overshoot, so the answer is as the same as in Problem 1(b), i.e. $\Phi_M = 53$. Now we need $\Phi_M = 63$. From the Bode plot, we find that the frequency at phase $\phi = -180 + 53 + 10 = 117$ is $\omega \approxeq 7.85$ rad/s.

(d) The magnitude at $\omega \approxeq 7.85$ rad/s is $20 \log |G(j7.85)| = 23.68$ dB. Then to shift the magnitude to 0 dB at this frequency we design the lag compensator to have high-frequency asymptote at -23.69 dB, and low frequency asymptote at 0 dB. We set the upper break frequency to 1 decade below $\omega = 7.85$, that is at $\omega_{up} = 0.785$. The difference between the upper and lower break points is -23.69 dB x $\dfrac{1\,\text{dec}}{-20dB} = 1.18$ dec, so the lower frequency is at $\omega = \frac{0.785}{10^1.18} = 0.0517$. So the compensator has the form $G_c(s) = \frac{K_c\,(s+0.785)}{s+0.0517}$. Since we want this to be 0 dB for low frequency, $|G_c\,(s=0)| = \left|\frac{K_c\,(0.785)}{0.0517}\right| = 1$. So the lag compensator is $G_c(s) = \frac{0.066\,(s+0.785)}{s+0.0517}$

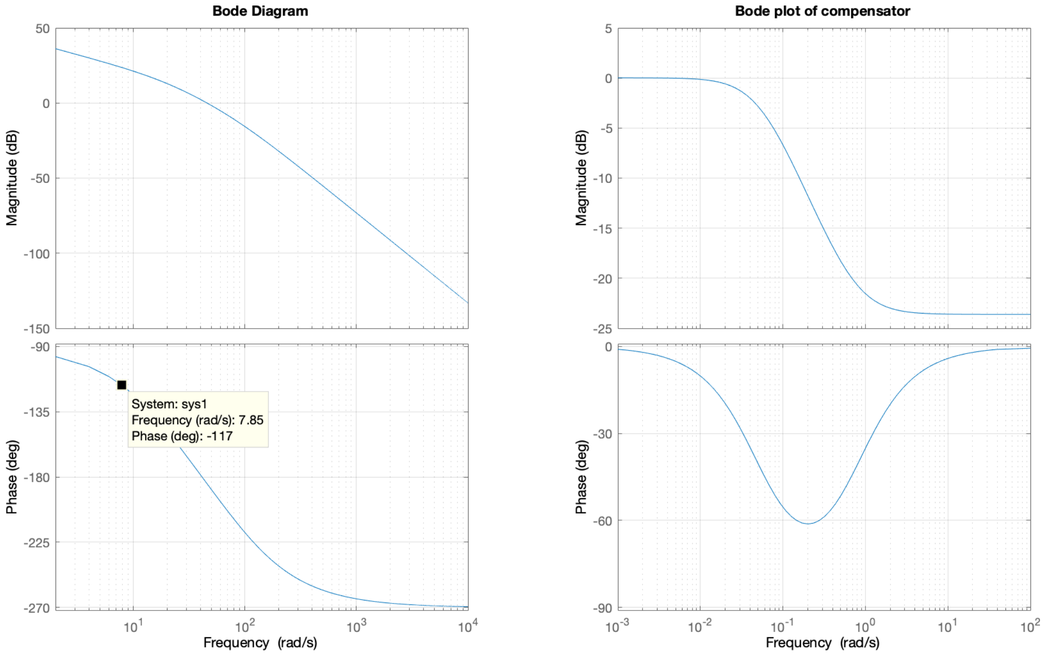

Figure 2: Bode plot for the original system and the designed compensator

(e) We increase the phase margin to account for the phase of the lag compensator. After $t = 20$ sec the error is $y(t) - t \approxeq 0.00772$, within the design specifications. The overshoot is 14 %. See Figure 3

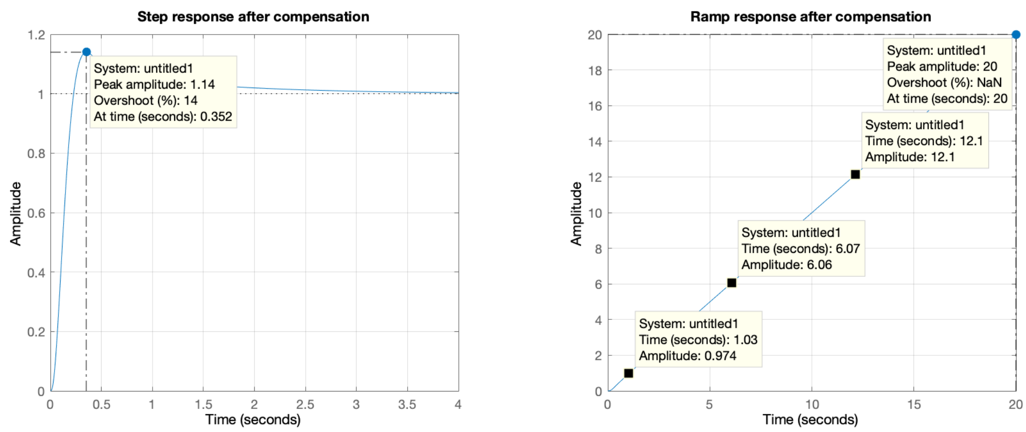

Figure 3: Step and ramp responses for the compensated plant

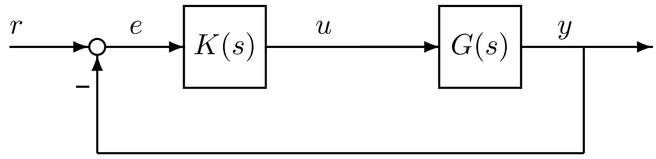

Figure 4: Feedback Loop

Problem 3. (10 points) Consider the loop in Figure 4 with $L(s) := G(s)K(s)$. Define two loop transfer functions:

(a) $L(s) = \frac{-8}{s+4}$ (b) $L(s) = \frac{2s+9}{s-4}$

Use the `nyquist` command in Matlab to generate the Nyquist plot for each loop transfer function. Then apply the Nyquist stability theorem to predict the number of closed-loop poles of the feedback system in the right-half plane.

Solution.
See the Nyquist plots in Figures 5 and 6. The Nyquist stability theorem says that if the number of RHP poles of the closed loop transfer function $\frac{L(s)}{1+L(s)}$ is Z, the number of RHP poles of the open loop transfer function $L(s)$ is P, and N is the number of clockwise rotations of the Nyquist plot around -1, then $Z = N + P$. In (a), $N = 1, P = 0$ so $Z = 1$. In (b), $N = -1, P = 1$ so $Z = 0$.

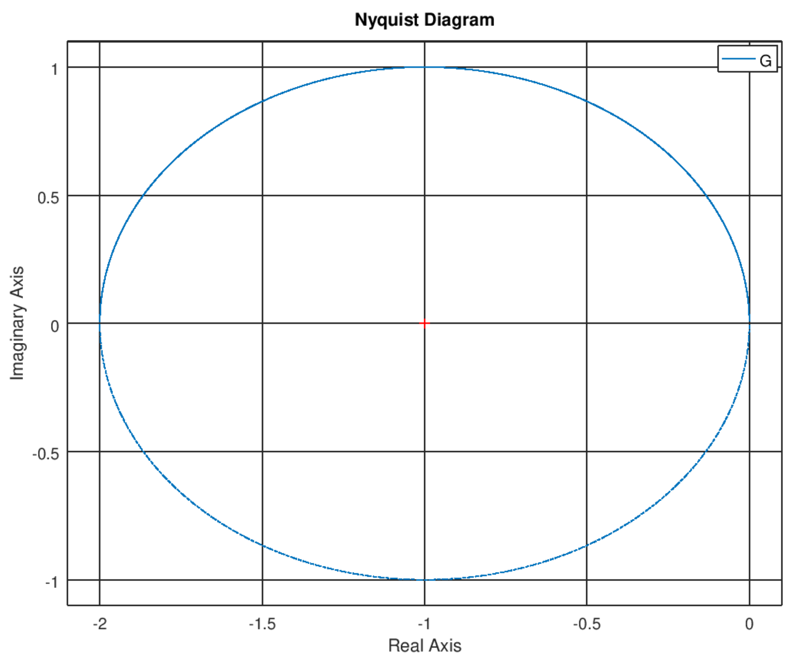

Figure 5: Nyquist Plot for 3(a)

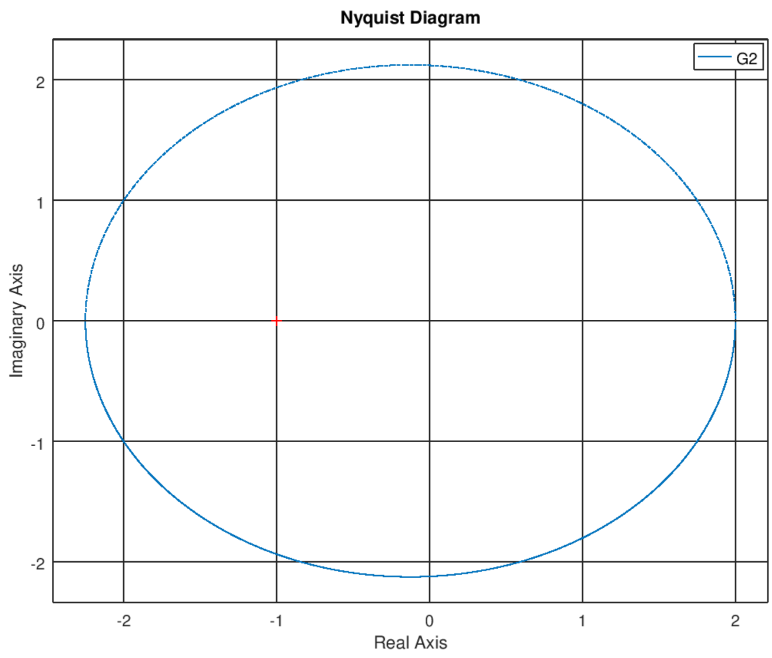

Figure 6: Nyquist Plot for 3(b)

Problem 4. (25 points) Consider the following:

$$G(s) = \frac{1}{(s+2)(s^2+2s+5)}$$

Suppose this is in unity feedback with a constant gain controller K. In other words, we have a negative feedback loop where the forward gain is $KG(s)$ and the loop gain is also $KG(s)$.

(a) Use the Routh-Hurwitz stability criterion to determine what values of K stabilize the closed-loop system.

(b) Sketch the Bode plots of $G(s)$ by hand.

(c) Verify your Bode plots using MATLAB.

(d) Using the Bode plot, sketch the Nyquist plot by hand. You should primarily use your sketch of the Bode plot, but you may use MATLAB to calculate exact numerical values as needed.

(e) Using the Nyquist plot, determine what values of K stabilize the closed-loop system. Does this match your answer from the Routh-Hurwitz criterion?

Note: In this problem, we have explicitly provided a step-by-step guide on how to use Nyquist plots to determine stability of a system. You are expected to be able to understand these principles and do this on your own, e.g. a problem on an exam may be "Use the Nyquist plots to determine the values of K which stabilize the closed-loop system." with no further guidance provided.

Solution.

(a) The characteristic eqution is: $s^3 + 4s^2 + 9s + 10 + K$ and Routh table reads as follows:

$$\begin{array}{lll} s^3 & 1 & 9 \\ s^2 & 4 & 10+K \\ s & -\frac{1}{4}(K-26) & 0 \\ s^0 & K+10 & \end{array}$$

which yields that $-10 < K < 26$.

(b) Skipped

(c) The plot is given below.

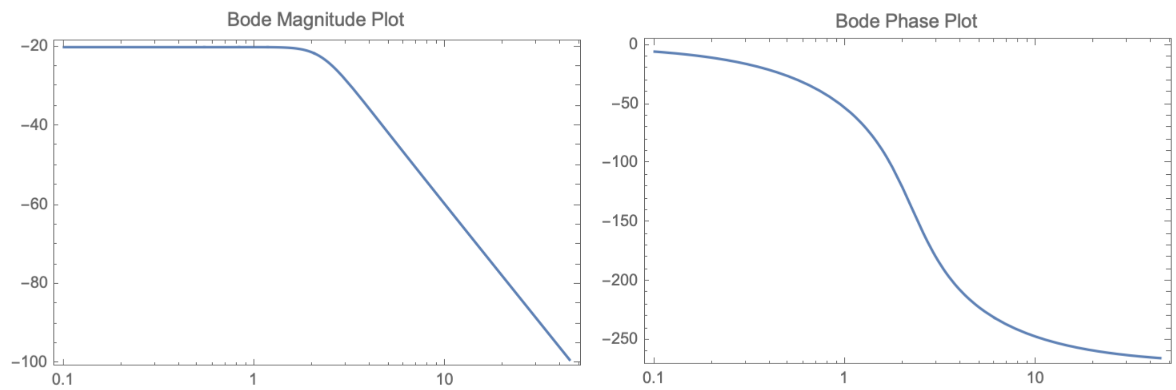

Figure 7: Bode plot of the given system

(d) The plot is given below:

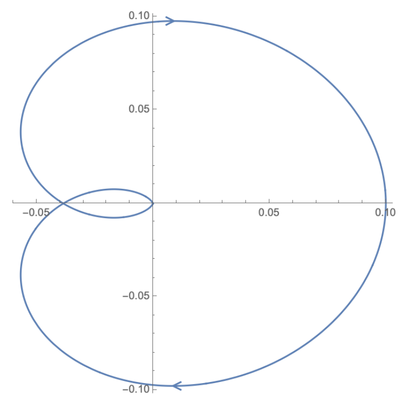

Figure 8: Nyquist plot of the given system

(e) Let N be the number of encirclements of $-1/K$ and P be the number of open loop poles. We have that $N = Z - P$ where Z is the number of closed loop poles. In this particular case $P = 0$ yielding $N = Z$. This gives that

$$-\frac{1}{K} > 10 \qquad \text{and} \qquad -\frac{1}{K} < -0.0384$$

which is the same condition as derived in part (a).

Problem 5. (15 points) Consider:

$$G(s) = \frac{1}{(s-1)(s+2)(s+4)} \qquad K = 10$$

(a) Sketch the Bode plot of $KG(s)$ by hand, and use the Bode plot to sketch the Nyquist plot.
(b) Use MATLAB to draw an exact Nyquist plot.
(c) Use this Nyquist plot to calculate the gain margin and phase margin.

Solution. (a) See the hand drawn Bode plot in Figure 9 and the hand drawn Nyquist plot in Figure 10. The important qualitative features of the Bode plot are:

- The magnitude decreases with increasing frequency.
- The phase starts at $-180°$, increases monotonically to a peak of around $-166°$, then decreases monotonically till it hits $-270°$.

We can draw the approximate Nyquist plot with just these observations in mind as given in Figure 10. The shape of the plot is exactly recovered, even though the precise lengths/sizes of the features couldn't be determined precisely. Since the important features, as far as the shape of the Nyquist plot is concerned, are the asymptotes of the phases and the one peak in the phase plot, the asymptotes will carry 0.75 points each and the peak carries 1 point. The Nyquist plot itself carries the remaining 2.5 points.

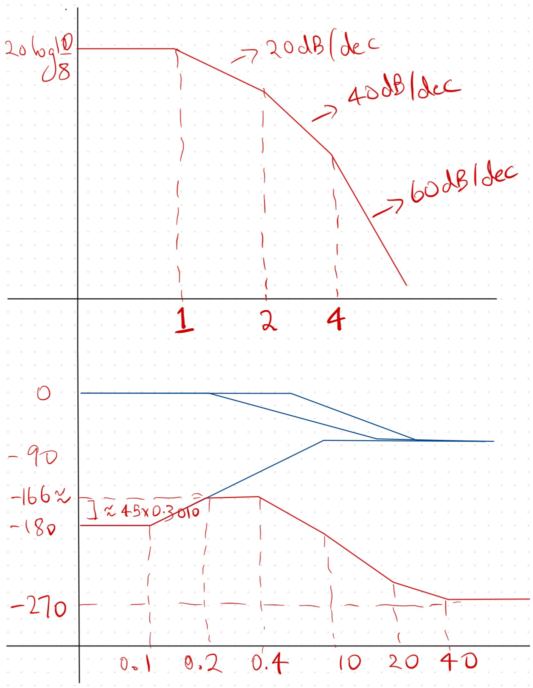

Figure 9: Approximate Bode plot of the given system

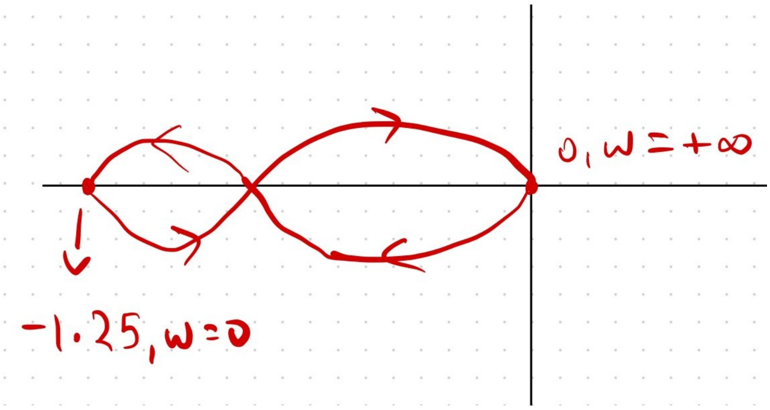

Figure 10: Approximate Nyquist plot of the given system

(b) Nyquist plot of $KG(s)$ is shown in Figure 11.

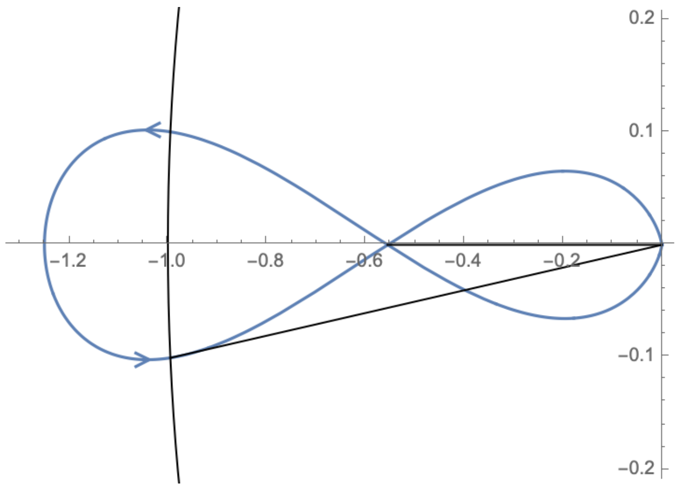

Figure 11: Nyquist plot of the given system

(c) From the Nyquist plot, two points (-0.55, -1.25) intersect the negative real axis. The corresponding gain margins are $1/0.55 \approx 1.8$ and $1/1.25 = 0.8$. The GM of the system is $[0.8, 1.8]$.

To obtain PM from the Nyquist plot, draw a unit circle and mark the points where unit circle intersects the Nyquist plot. Draw a line from the origin to one of the points. The angle formed between that line and the $-180°$ axis is the PM. The Nyquist plot here intersects the unit circle at two symmetric points, one of which is $\approx (-0.995, -0.1)$ and so the PM is $\arctan\left(\dfrac{-0.1}{-0.995}\right) = 5.74$ degrees.