Problem 1. (40 points) For each of the transfer functions given below, draw the Bode plots (both magnitude and phase) by hand, using the techniques discussed in Lecture 15. Explain all steps in your drawing procedures.

(a) $L(s) = \dfrac{s+8}{s(s+4)}$

(b) $L(s) = \dfrac{8s}{s^2+0.2s+4}$

(c) $L(s) = \dfrac{s^2+0.2s+1}{s(s+0.2)(s+6)}$

(d) $L(s) = \dfrac{s+10}{s(s^2+1.4s+1)}$

After you're done, check your results using MATLAB. (Note that the bode command in MATLAB plots magnitude in decibels.) Turn in both the hand sketches and the MATLAB plots.

Solution.

(a) We can break the given transfer function into its constituents and by applying what we know of the transfer functions of Type 1, Type 2, etc. we get the following plot:

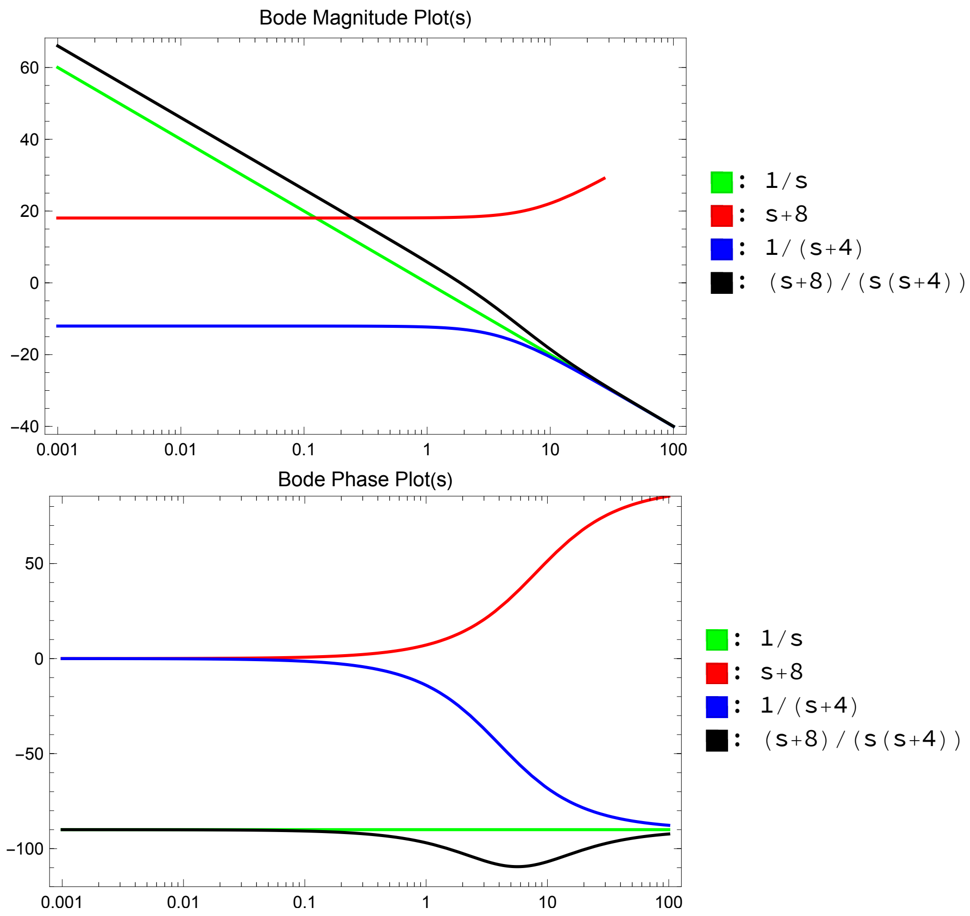

Figure 1: The Bode plots for part (a).

(b) We follow a similar approach in this case, and keep the second order part in its standard form so we can directly use it. Therefore we have,

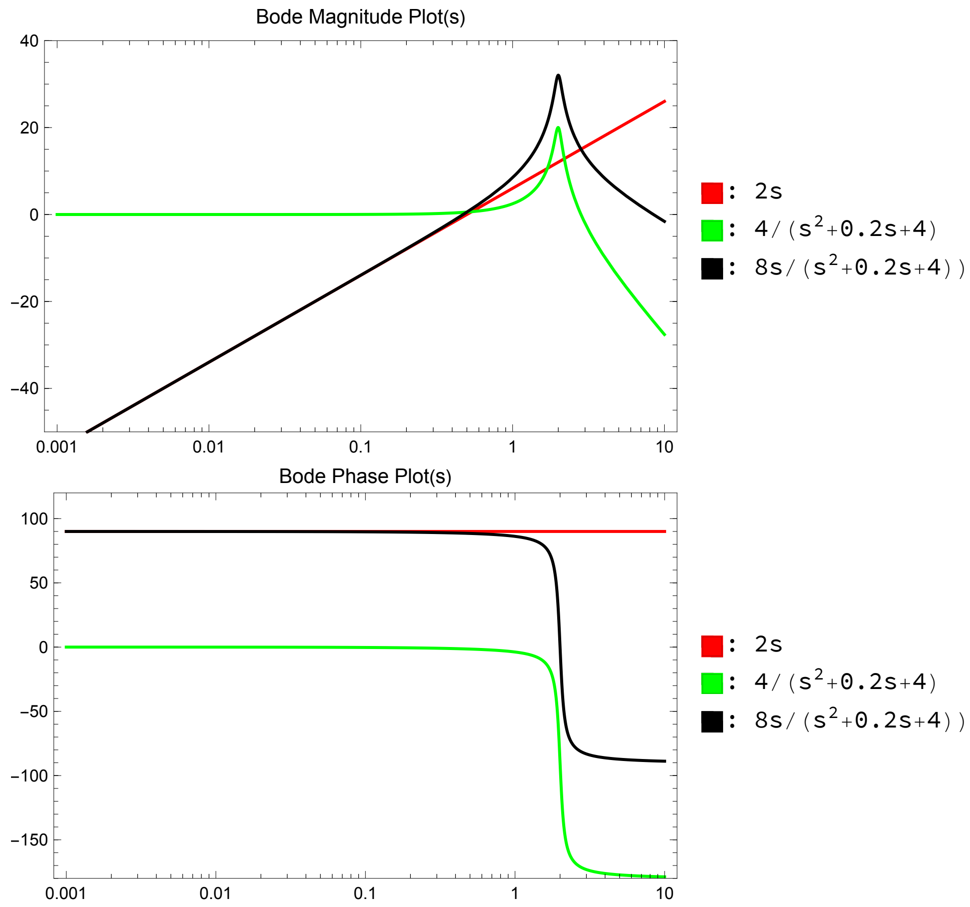

Figure 2: The Bode plots for part (b).

(c) Similarly,

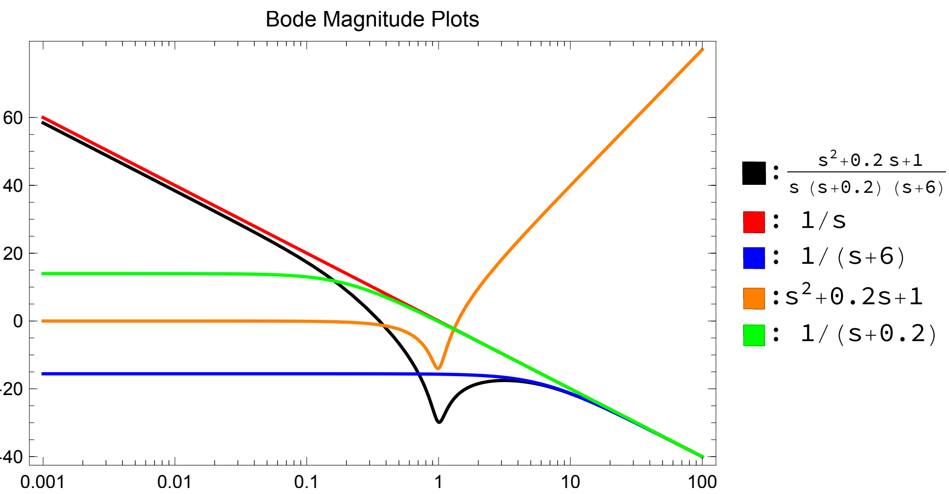

Figure 3: The Bode plots for part (c), magnitude.

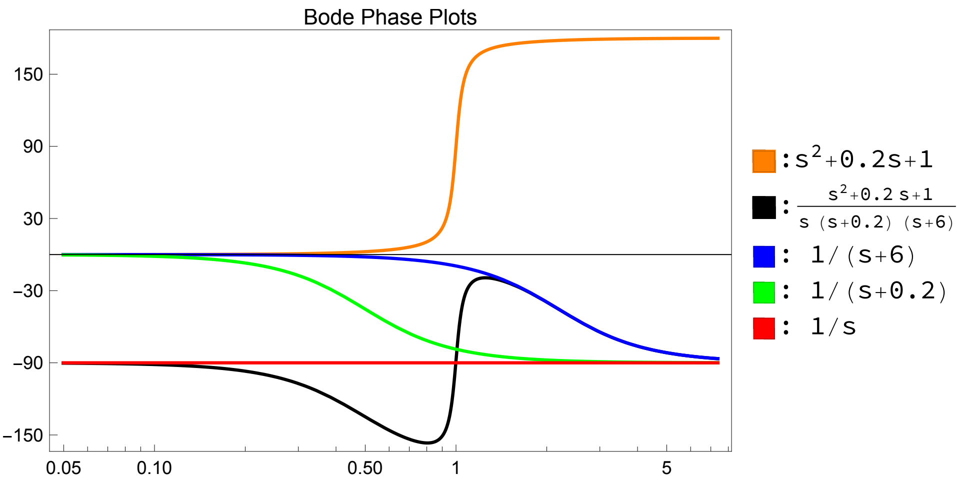

Figure 4: The Bode plots for part (c), phase.

(d) Finally,

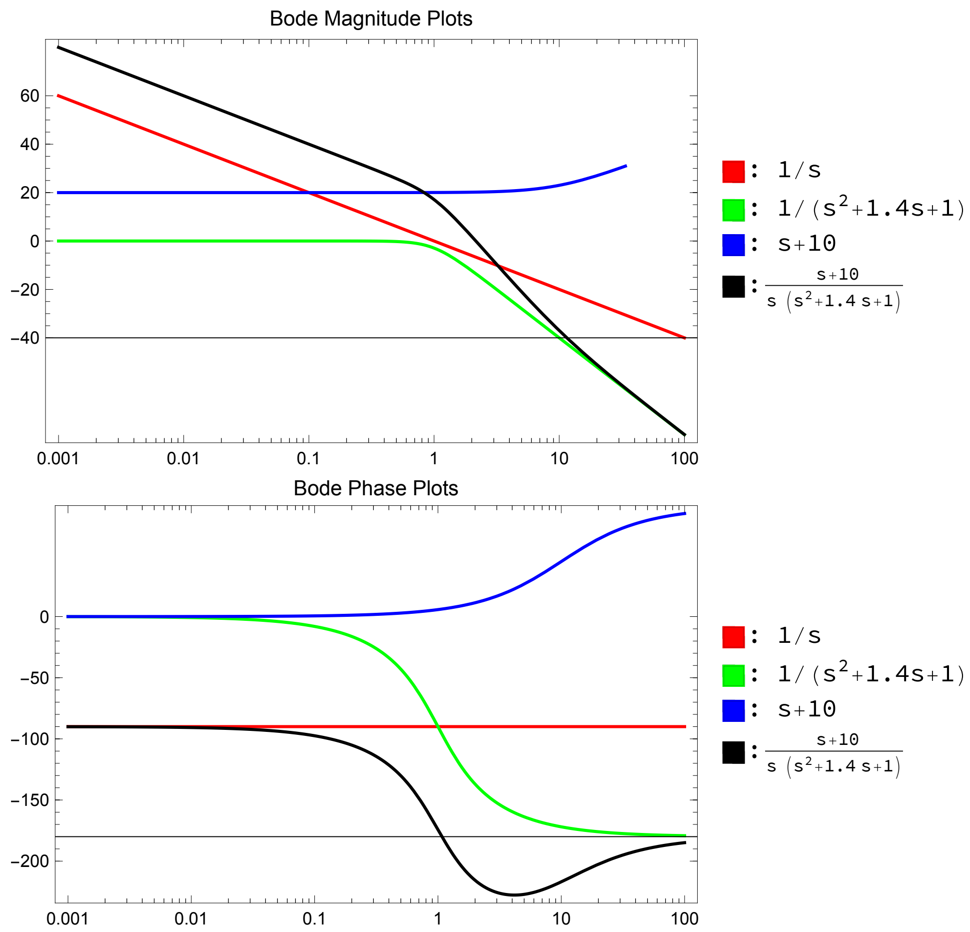

Figure 5: The Bode plots for part (d).

Problem 2. Consider the feedback diagram in Figure 6 below. Suppose $L(s) := G(s)K(s)$. Consider two pairs of plants and controllers:

i) $K(s) = \dfrac{10(s+3)}{s}$ and $G(s) = \dfrac{-0.5(s^2 - 2500)}{(s-3)(s^2+50s+1000)}$

ii) $K(s) = \dfrac{0.4s+1}{s}$ and $G(s) = \dfrac{1}{s+1}$

iii) $K(s) = 2$ and $G(s) = \dfrac{1}{(s+1)^3}$

Perform the following calculations for each plant/controller pair:

(a) (15 points) Verify that the feedback system is stable.
(b) (15 points) Use the Bode plot of $L(s)$ to compute the gain margin(s) of the feedback system.
(c) (15 points) Use the Bode plot of $L(s)$ to figure out the phase margins of the feedback system.

You can use the `allmargin` command to check your answers.

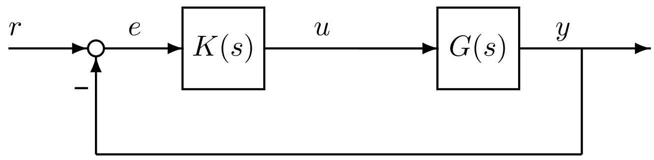

Figure 6: Feedback Loop

Solution.

(i) For this system we have that the feedback system is indeed stable as per the Routh Array below. Note that the characteristic equation is:

$$s^4 + 42s^3 + 835s^2 + 9500s + 37500 = 0$$

(a) Which gives,

s^4	1	835	37500
s^3	42	9500	0
s^2	608.8	37500	0
s	6912.9	0	0
s^0	37500	0	0

(b) The gain margins are computed at the frequencies when phase is $-\pi$. This happens at frequencies of 3.59 rad/s and 26.38 rad/s. The corresponding gain margins are 0.287 and 0.235.

(c) The phase margins are computed when the Bode magnitude (dB) plot is zero. For this system we can compute it from the Bode plot as 26.34°. The Bode plot is given below in Figure 7.

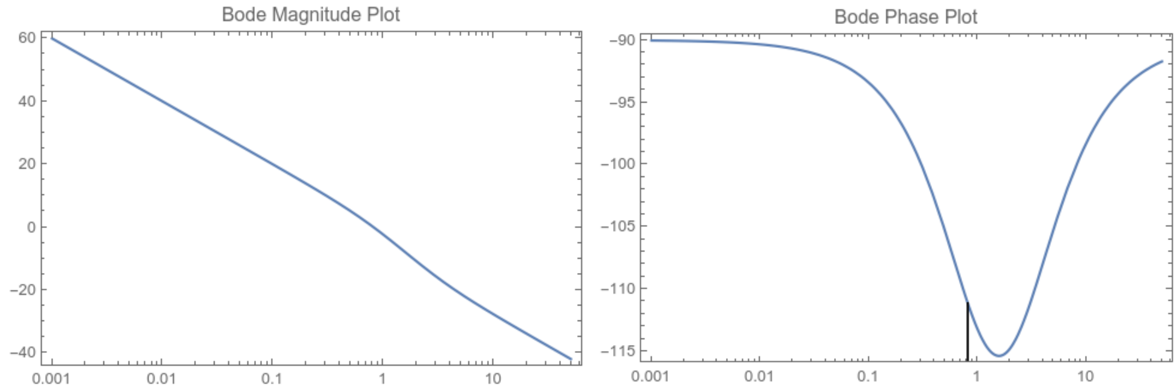

Figure 8: Bode plot for the second system

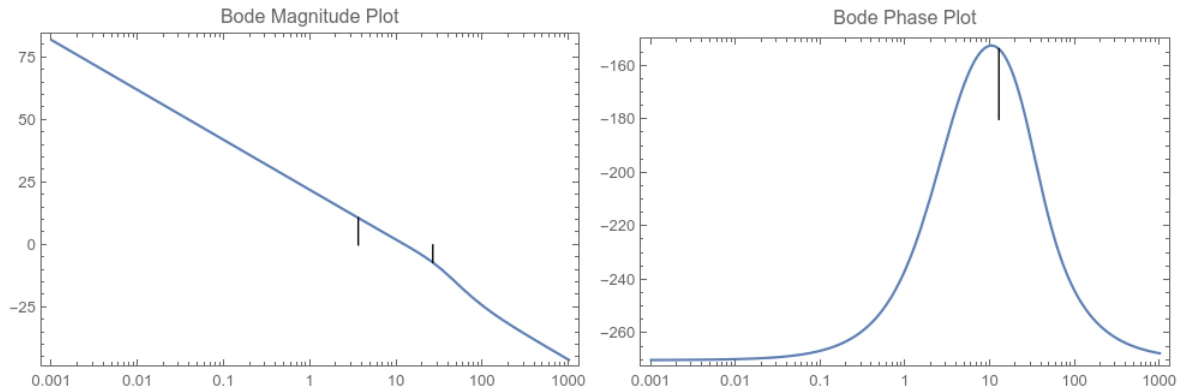

Figure 7: Bode Plot for the first system

(ii) For the second system we have the characteristic equation is

$$5s^2 + 7s + 5 = 0 \qquad \Longrightarrow \qquad s^2 + \frac{14s}{10} + 1 = 0$$

(a) Which gives us the Routh Array,

$$\begin{array}{lll} s^2 & 1 & 1 \\ s & \frac{7}{5} & 0 \\ s^0 & 1 & 10 \end{array}$$

which shows that the system is stable.

(b) There is no gain margin since there is no frequency at which the phase is $-180°$.

(c) We find that this system has a phase margin of $68.87°$ at a frequency of 0.815 rad/s. The Bode plot is given in Figure 8

(iii) For the third system, the characteristic equation is

$$s^3 + 3s^2 + 3s + 3 = 0$$

(a) Which gives us the Routh Array,

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 3 & 3 \\ s^1 & 2 & 0 \\ s^0 & 3 & 0 \end{array}$$

which shows that the system is stable. The gain margins are computed at the frequencies when phase is $-\pi$. This happens at frequencies of 1.7322 rad/s. The corresponding gain margins is 4.0006.

(b) The phase margins are computed when the Bode magnitude (dB) plot is zero. For this system we can compute it from the Bode plot as 67.6058°. The Bode plot is given in Figure 9

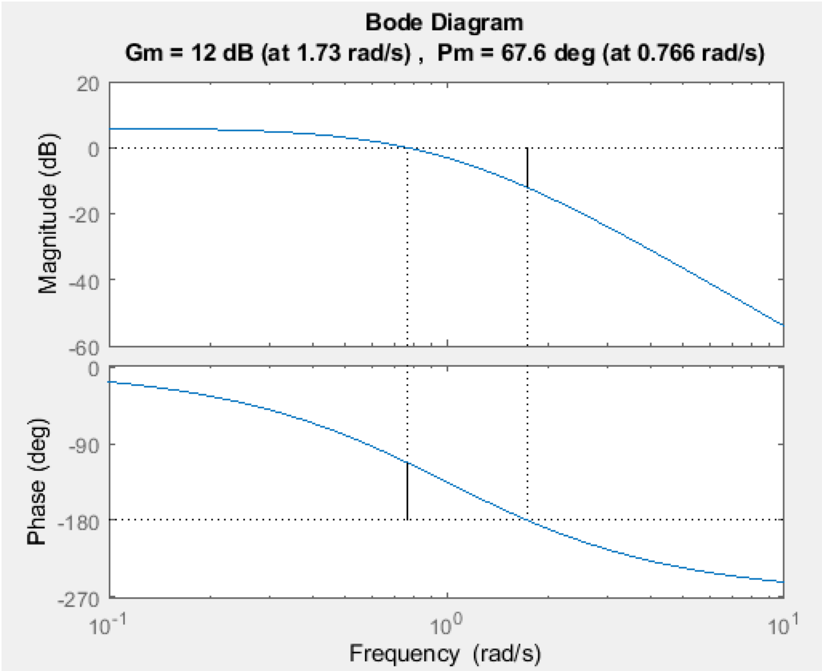

Figure 9: Bode plot for the second system

Problem 3. (15 points) Show that for the transfer function $KG(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$, the phase margin is independent of ω_n and is given as $\tan^{-1}\left(\dfrac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}\right)$.

Solution.
Given,

$$KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

To calculate the phase margin, we first find the gain-crossover-frequency (ω_c):

$$\left|KG(j\omega)\right|\Big|_{\omega=\omega_c} = 1 \Rightarrow \frac{\omega_n^2}{|-\omega_c^2 + 2j\omega_n\omega_c\zeta|} = 1$$
$$\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_c^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1$$

Therefore,

$$\omega_n^4 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2$$
$$\Leftrightarrow \omega_n^4 + \left(2\zeta^2\omega_n^2\right)^2 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 + \left(2\zeta^2\omega_n^2\right)^2$$
$$\Leftrightarrow \omega_n^4\left(1+4\zeta^4\right) = \left(\omega_c^2 + 2\zeta^2\omega_n^2\right)^2$$

Thus, excluding the negative root (why?),

$$\begin{aligned} \omega_c^2 &= -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4+1} \\ &= \left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)\omega_n^2 \end{aligned}$$

So we get,

$$KG(j\omega) = \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c} \qquad \text{and} \qquad \angle KG(j\omega) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega_c}{-\omega_c^2}\right)$$

which is

$$\tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)$$

Note that $\theta = \tan^{-1} x \iff \pi + \theta = \tan^{-1}(x)$.