Problem 1. (20 points) Answer the following questions.

(a) Consider the following system $G(s)$ and sinusoidal input:

$$-3\dot{y}(t) - 2y(t) = 7u(t)$$
$$u(t) = 6\cos(t+4)$$

What is the magnitude and phase of $G(1j)$? Is the steady-state output bounded? If yes, what is it?

(b) Consider the following system $G(s)$ and sinusoidal input:

$$\ddot{y}(t) + 0.1\dot{y}(t) + 4y(t) = \dot{u}(t) + 2u(t)$$
$$u(t) = -\cos(2t)$$

What is the magnitude and phase of $G(2j)$? Is the steady-state output bounded? If yes, what is it?

Solution.
For a sinusoidal input $u(t) = A\cos(\omega t)$ an LTI system G with zero initial conditions, the system output can be expressed as follows:

$$y(t) = A\,|G(j\omega)|\cos(\omega t + \angle G(j\omega))$$

(a) We derive the transfer function as $G(s) = \frac{-7}{3s+2}$. This gives $|G(j)| = \frac{7\sqrt{13}}{13}$ and $\angle G(j) = \arctan\left(\frac{3}{-2}\right) = \pi - \arctan(3/2) = 123.69°$. The steady state output is bounded and can be readily obtained:

$$y_{ss}(t) = 6|G(j)|\cos(t + 4 + \angle G(j)) \approx 11.65\cos(t + 61.6) \quad (1)$$

(b) Here the transfer function is given by

$$G(s) = \frac{s+2}{s^2 + s/10 + 4}$$

and so $|G(2j)| = 10\sqrt{2}$ and $\angle G(2j) = -\pi/4$. Again, the steady state output is bounded and given by:

$$y_{ss}(t) = -10\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right) \quad (2)$$

Problem 2. (15 points) Figure 1 shows an input $u(t)$ and the corresponding output $y(t)$ generated by a linear system $G(s)$. The input has the form $u(t) = A_0\cos(\omega_0 t)$.

(a) What are the values of A_0 and ω_0 for the input signal?
(b) What is the magnitude $|G(j\omega_0)|$?
(c) What is the phase $\angle G(j\omega_0)$ in degrees?

Solution.

(a) From the blue waveform $A_0 = 2$ and $\omega_0 = \frac{2\pi}{80-64} = \frac{\pi}{8}$.

(b) Comparing with the output wave, we get $|G(j\omega_0)| = \frac{6}{A_0} = 3$.

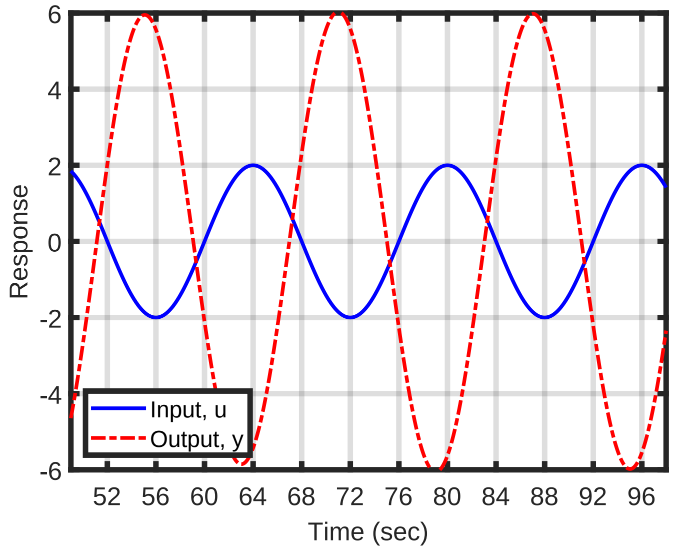

Figure 1: Input $u(t)$ and output response $y(t)$ for system $G(s)$.

(c) Comparing their periods we have that $\angle G(j\omega_0) = -\frac{7}{16} \times 2\pi = -\frac{7\pi}{8} \approxeq -157.5°$.

Problem 3. (40 points) Sketch the Bode plots by hand for the following systems:

(a) A PI controller with input $e(t)$ and output $u(t)$:

$$\dot{u}(t) = K_p\dot{e}(t) + K_i e(t)$$

with $K_p = 10$ and $K_i = 1$.

(b) A "low frequency boost" controller with input $e(t)$ and output $u(t)$:

$$\dot{u}(t) + u(t) = \dot{e}(t) + 10e(t)$$

This type of controller will be encountered later in the course.

(c) A first-order system with right-half plane zero with input $u(t)$ and output $y(t)$:

$$2\dot{y}(t) + 0.6y(t) = -\dot{u}(t) + 30u(t)$$

(d) A second-order underdamped system with input $u(t)$ and output $y(t)$:

$$\ddot{y}(t) + 0.2\dot{y}(t) + 4.01y(t) = -u(t)$$

After you're done, check your results using MATLAB. Turn in both the hand sketches and the MATLAB plots.

Solution.

(a) Take the Laplace transform to yield the transfer function:

$$sU = 10sE + E \qquad \Longrightarrow \qquad G(s) = \frac{10s + 1}{s}$$

which in Bode form can be written as:

$$G(j\omega) = \frac{10j\omega + 1}{j\omega}$$

Now,

- **Type 1:** We have $n = -1$ and $K_0 = 1$.
- **Type 2:** There is one term with $10j\omega + 1$.

The Bode plot is given in Figure 2.

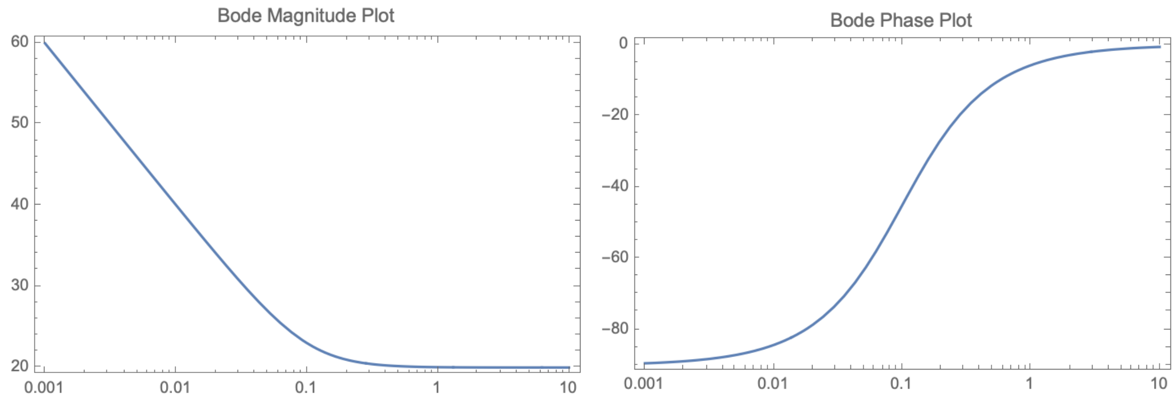

Figure 2: Bode plot for part (a) of the question.

(b) Again, take the Laplace transform to get:

$$sU + U = sE + 10E \qquad \Longrightarrow \qquad G(s) = \frac{s+10}{s+1}$$

which in Bode form can be written as:

$$G(s) = 10\frac{\frac{s}{10}+1}{s+1} \Longrightarrow G(j\omega) = 10\frac{\frac{j\omega}{10}+1}{j\omega+1}$$

. Now,

- **Type 1:** We have $n = 0$ and $K_0 = 10$.
- **Type 2:** There are two terms with

$$\diamond \quad \frac{j\omega}{10} + 1 \qquad \text{and} \qquad \diamond \quad j\omega + 1$$

The Bode plot is given in Figure 3.

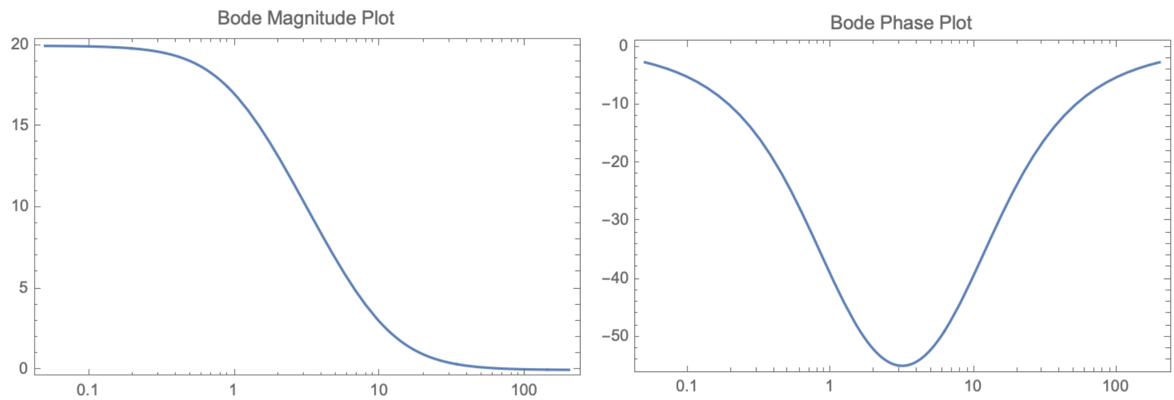

Figure 3: Bode plot for part (b) of the question.

(c) Now the Laplace transform yields the transfer function:

$$2sY + 0.6Y = -sU + 30U \qquad \Longrightarrow \qquad G(s) = \frac{30-s}{2s+0.6}$$

which in Bode form can be written as:

$$G(s) = 50\frac{\frac{-1}{30}s + 1}{\frac{10}{3}s + 1} = 50\frac{\frac{-1}{30}j\omega + 1}{\frac{10}{3}j\omega + 1}$$

Now,

- **Type 1:** We have $n = 0$.
- **Type 2:** There are two terms with

$\diamond \ \frac{-1}{30}j\omega + 1$ and $\diamond \ \frac{10}{3}j\omega + 1$

The Bode plot is given in Figure 4.

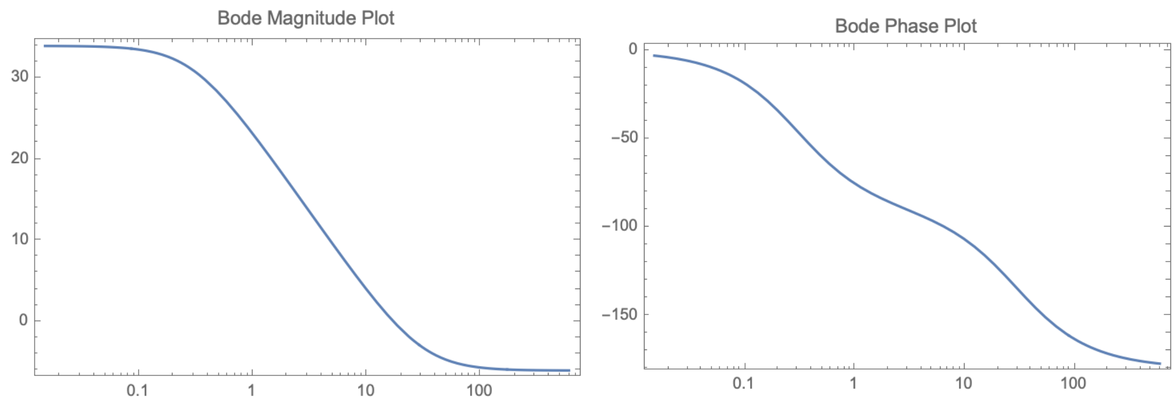

Figure 4: Bode plot for part (c) of the question.

(d) Finally, take the Laplace transform to get:

$$s^2Y + 0.2sY + 4.01Y = -U \qquad \Longrightarrow \qquad G(s) = \frac{-1}{s^2 + 0.2s + 4.01}$$

which in Bode form is written as:

$$G(s) = \frac{-1}{4.01}\frac{1}{\frac{s^2}{4.01} + \frac{0.2}{4.01}s + 1} \Longrightarrow G(j\omega) = \frac{-1}{4.01}\frac{1}{\frac{(j\omega)^2}{4.01} + \frac{0.2}{4.01}j\omega + 1}$$

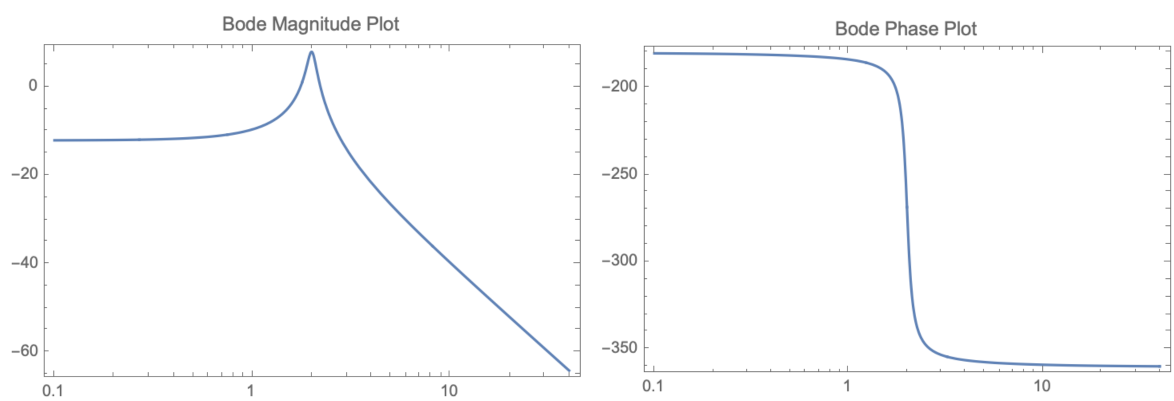

Figure 5: Bode plot for part (d) of the question.

Problem 4. Now we consider Bode plots for higher-order systems.

(a) (10 points) Consider a feedback loop with the following plant $G(s)$ and PI controller $K(s)$:

$$\dot{y}(t) + 2y(t) = 3u(t)$$
$$u(t) = 10e(t) + 2\int_0^t e(\tau)\, d\tau$$

Sketch the Bode plots of: $G(s)$, $K(s)$, and the product $G(s)K(s)$. What are some of the differences between the Bode plots of $G(s)$ and $G(s)K(s)$?

(b) (15 points) Consider a feedback loop with the following plant $G(s)$ and PD controller $K(s)$:

$$\ddot{y}(t) + 0.4\dot{y}(t) + 4y(t) = 8u(t)$$
$$u(t) = e(t) + 0.5\dot{e}(t)$$

Sketch the Bode plots of: $G(s)$, $K(s)$, and the product $G(s)K(s)$. What are some of the differences between the Bode plots of $G(s)$ and $G(s)K(s)$?

You may check your answers using `Matlab`. However you should turn in sketches of each Bode plot by hand (not computer generated plots).

Solution.

(a) The bode plots of $G(s)$, $K(s)$, and $G(s)K(s)$ can be found in Figure 6. Key differences: (i) Magnitude: The behavior changes only in the low frequency regime where G had a constant magnitude while GK decreases at $20dB/dec$. This means that $|S(j\omega)| << 1$ at low frequencies, and much more so with the controller. This leads to better reference tracking and disturbance rejection. (ii) Phase: The gain crossover frequency gets pushed further to the right by K. Combining this with the observation that the phase plot behaviors at high frequency remain the same, decreasing towards -90^o, we should expect a lower phase margin after adding the controller.

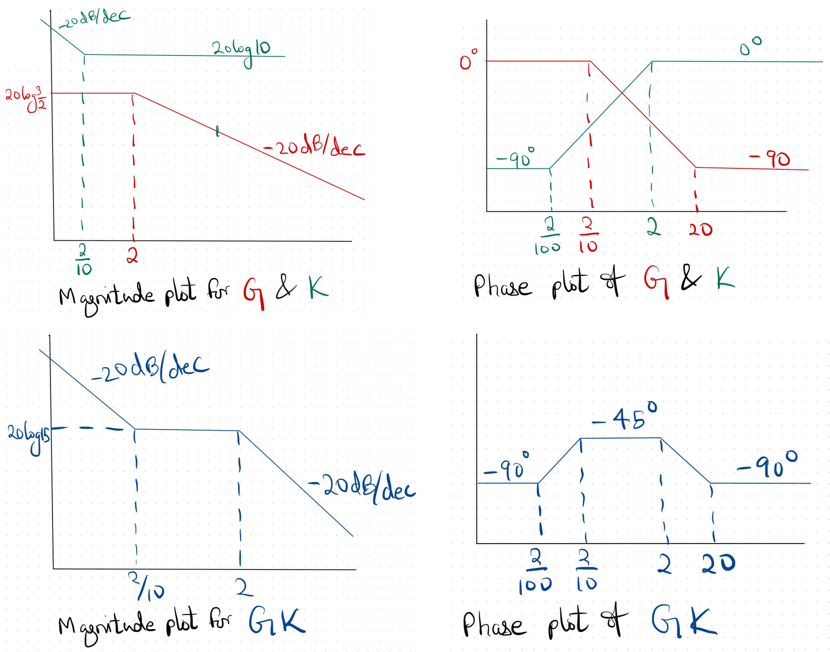

Figure 6: Bode plot for part (a) of the question.

(b) The bode plots of $G(s)$, $K(s)$, and $G(s)K(s)$ can be found in Figure 7. Key differences: (i) Magnitude: The only difference in magnitude is at high frequencies where the addition of the controller leads to a less sharp roll off. This means the addition of the PD controller can negatively affect the noise rejection capability of the system as $|T(j\omega)|$ should be really at higher frequencies for better noise rejection. (ii) The phase plot is pushed much above 180^o which should lead to better phase margin after the addition of the controller.

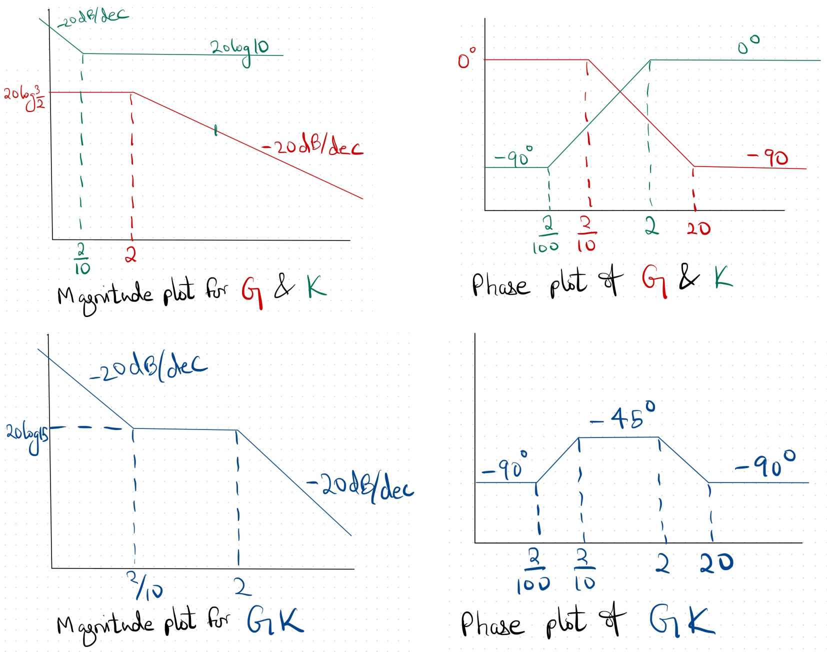

Figure 7: Bode plot for part (b) of the question.

The purpose of this exercise was to show the power of Bode plots as a tool for control design. One could read off a lot of important properties of the closed loop system simply by looking at the Bode plots of the open loop system. In addition, the effect of adding a controller in the loop on the Bode plot could be easily understood by just adding lines by hand. For instance, the addition of a PI controllers in (a) lead to better reference tracking and disturbance rejection but more overshoot (lower PM).