Problem 1. (15 points) Consider the following plant:

$$\dot{y} + 5y = 3u$$

Suppose a PI controller is used with P gain K_p and I gain K_i. Assume the closed-loop system is stable. Then what sampling time would you recommend for a discrete-time implementation? Write out Δt as a function of K_p and K_i.

Solution.
Since we are using PI control, $u(t) = k_p e(t) + K_i \int_0^t e(\tau)d\tau$ with $e(t) = r(t) - y(t)$. The closed-loop system can be expressed as:

$$\ddot{y} + (5 + 3K_p)\dot{y} + 3K_i y = 3K_p \dot{r} + 3K_i r.$$

Performing Laplace transform on both sides, we have the closed-loop transfer function

$$T(s) = \frac{3K_p s + 3K_i}{s^2 + (5 + 3K_p)s + 3K_i}.$$

The poles of the closed-loop system is

$$s_{1,2} = \frac{-(5 + 3K_p) \pm \sqrt{(5 + 3K_p)^2 - 12K_i}}{2}.$$

Now we need to consider the following two scenarios:

(i) If $(5 + 3K_p)^2 - 12K_i \geq 0$, s_1 and s_2 are both real, the corresponding time constant is
$\tau_1 = \dfrac{2}{|-(5 + 3K_p) + \sqrt{(5 + 3K_p)^2 - 12K_i}|}$, $\tau_2 = \dfrac{2}{(5 + 3K_p) + \sqrt{(5 + 3K_p)^2 - 12K_i}}$. For
τ_1, we have
$$\Delta t_1 = \frac{1}{5|(5 + 3K_p) - \sqrt{(5 + 3K_p)^2 - 12K_i}|},$$

For τ_2, we have
$$\Delta t_2 = \frac{1}{5(5 + 3K_p) + 5\sqrt{(5 + 3K_p)^2 - 12K_i}}.$$

Since $\Delta t_2 \leq \Delta t_1$, we can choose $\Delta t = \Delta t_2$.

(ii) If $(5 + 3K_p)^2 - 12K_i \geq 0$, s_1 and s_2 are a complex conjugate, which has same real part and
hence same time constant $\tau = \dfrac{2}{5 + 3K_p}$. In this case, we should design $\Delta t = \dfrac{1}{10}\dfrac{2}{5 + 3K_p} = \dfrac{1}{25 + 15K_p}$.

Problem 2. (40 points) Sketch the root loci for the following $L(s)$ by hand by applying rules A–F. (Recall that the root locus plots how the solutions of $1 + KL(s) = 0$ vary as K goes from 0 to $+\infty$.

(a) $L(s) = \dfrac{1}{s^2 + 2s + 20}$ (c) $L(s) = \dfrac{(s + 1)(s + 2)}{s(s^2 + 4)(s^2 + 5)}$

(b) $L(s) = \dfrac{s - 3}{s^2 + 2s + 20}$ (d) $L(s) = \dfrac{s + 3}{s^5 + 1}$

Solution.
The application of Rules A-F for the given systems results in the following table: The required plots are shown in Figure 1.

Rule / System	System (a)	System (b)	System (c)	System (d)
A (# of branches)	2	2	5	5
B (branch starts)	$-1 \pm \sqrt{19}j$	$-1 \pm \sqrt{19}j$	$0, \pm 2j, \pm\sqrt{5}j$	$-1, \pm\sqrt{2}/2, \pm\sqrt{2}j/2$
C (branch ends)	∞	$3, \pm\infty$	$-1, -2, \infty$	$-3, \pm\infty$
D (real RL)	None	$(-\infty, 3)$	$(-\infty, -2) \cup (-1, 0)$	$(-3, -1)$
E (exit angles)	$90°, 270°$	$180°$	$2\pi k/3, k = 1, 2, 3$	$\pi k/4, k = 1, 3, 5, 7$
F ($j\omega$-crossings)	None	$\omega = 0$	None	None

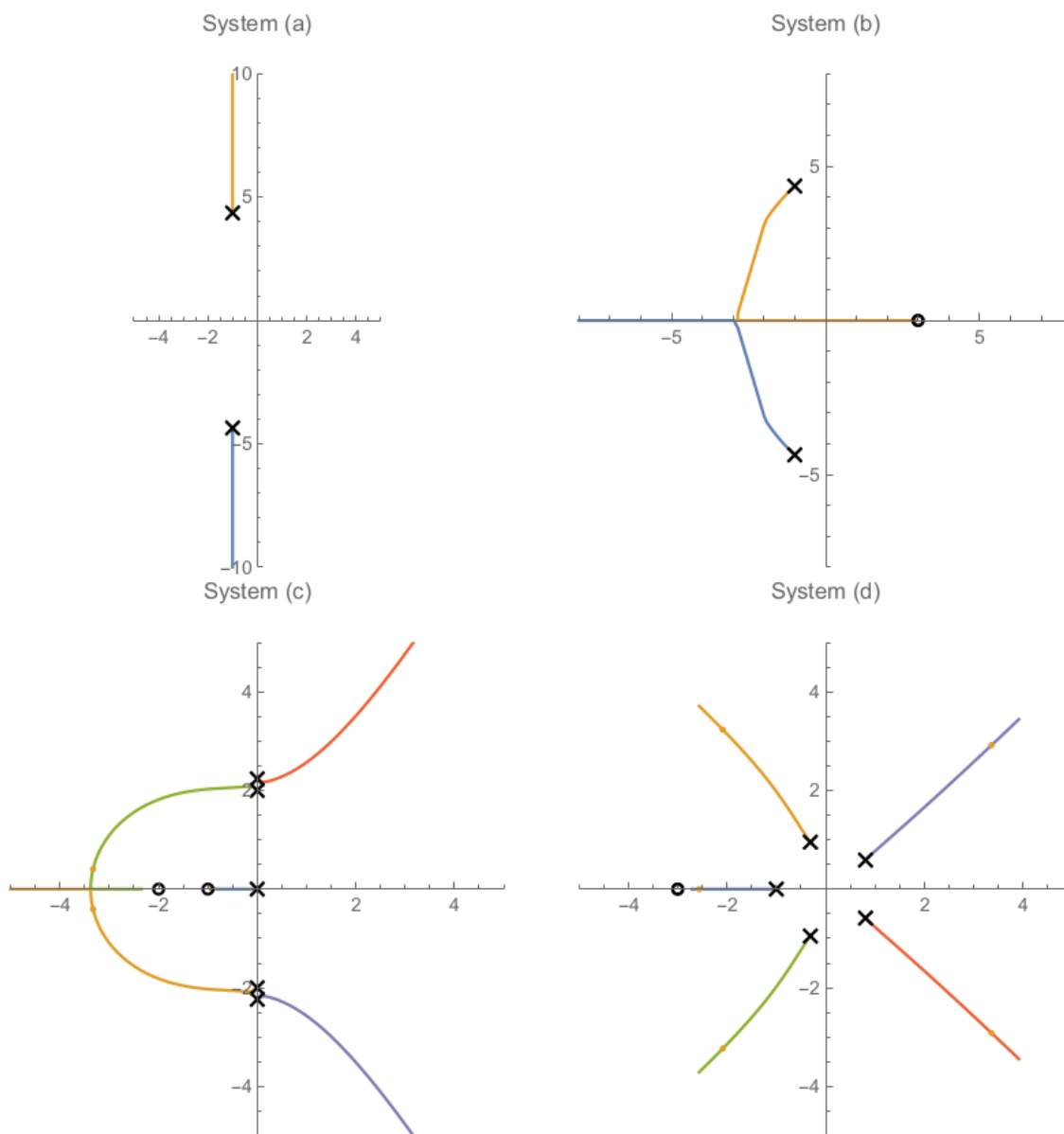Table 1: Results obtained from application of root locus rules

Figure 1: The root-locus plots for the given systems

Problem 3. (15 points) Consider the feedback system in Figure 2. Determine the transfer function for $Y(s)/R(s)$, and write the characteristic equation in terms of K. In other words, find the polynomials $a(s)$ and $b(s)$ such that the closed loop poles are the values of s that satisfy to $a(s) + Kb(s) = 0$.

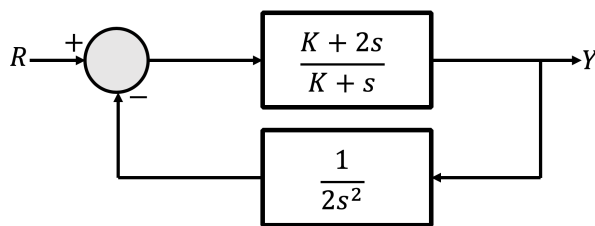You do **not** have to do this for this problem, but you should understand how to draw a root

Figure 2: A feedback system.

locus for where the closed loop pole locations as K goes from 0 to $+\infty$. (This shows that the root locus methods apply to more systems than the constant-gain unity-feedback setting discussed in class.)

Solution.
We know that for a plant $H(s)$ in negative feedback with a controller $G(s)$ the closed loop transfer function is given by:

$$\frac{H(s)}{1 + H(s)G(S)}$$

Here,

$$H(s) = \frac{K + 2s}{K + s}, \qquad \text{and} \qquad G(s) = \frac{1}{2s^2}$$

Therefore,

$$\frac{Y(s)}{R(s)} = \frac{2s^2(K + 2s)}{2s^2(K + s) + K + 2s}$$

This gives us that the characteristic equation:

$$2Ks^2 + 2s^3 + K + 2s = 0$$
$$\implies 2s^3 + 2s + K(2s^2 + 1) = 0$$

with $a(s) = 2s^3 + 2s$ and $b(s) = 2s^2 + 1$. Therefore in the form $1 + KL(s)$ we have that $L(s) = \frac{b(s)}{a(s)}$ yielding the root-locus on the left in Figure 3.

Problem 4. (30 points) Suppose the closed loop transfer function is given by:

$$\frac{KL(s)}{1 + KL(s)}$$

where K is some constant control gain.

(a) If $L(s)$ has 3 LHP poles and 1 LHP zero, is the closed-loop system stable for very large values of $K > 0$?

(b) If $L(s)$ has 2 LHP poles, 1 RHP poles, and 3 LHP zeros, is the closed-loop system stable for very large values of $K > 0$?

(c) If $L(s)$ has 5 LHP poles, 4 LHP zeros, and 1 RHP zeroes, is the closed-loop system stable for very large values of $K > 0$?

Give detailed justification for each answer.

Solution.

(a) Although, all poles and zeros are in the LHP, it does not necessary mean that all values of gain, K, will keep the system stable. Suppose that the transfer function is strictly proper, then the asymptotes should be taken into consideration. If there are $j\omega$-crossings, then some asymptotes may escape into RHP with high gain, causing the system to be unstable. For example, consider a system with a zero at $s = -5$ and poles at $s = -2, s = -1 \pm j$. Although, it has 3 LHP poles and 1 LHP zero, the root locus plot of this system shows that there are $j\omega$-crossings and the system becomes unstable with high gain.

(b) Root locus branches start at the poles and end at the zeros. If the number of poles is greater than the number of zeros, then some branch(es) will go to infinity. In this case, there is one unstable pole (RHP pole). Since there are equal numbers of poles and zeros, all branches will end at the zeros and because all the zeros are in the LHP, a large enough K can stabilize the closed-loop system.

(c) Similar to the previous problem, all branches will start at the poles and end at the zeros. This case also has equal number of both. However, one of the zeros is in the RHP. Therefore, a large value of K can cause the closed-loop system to be unstable.
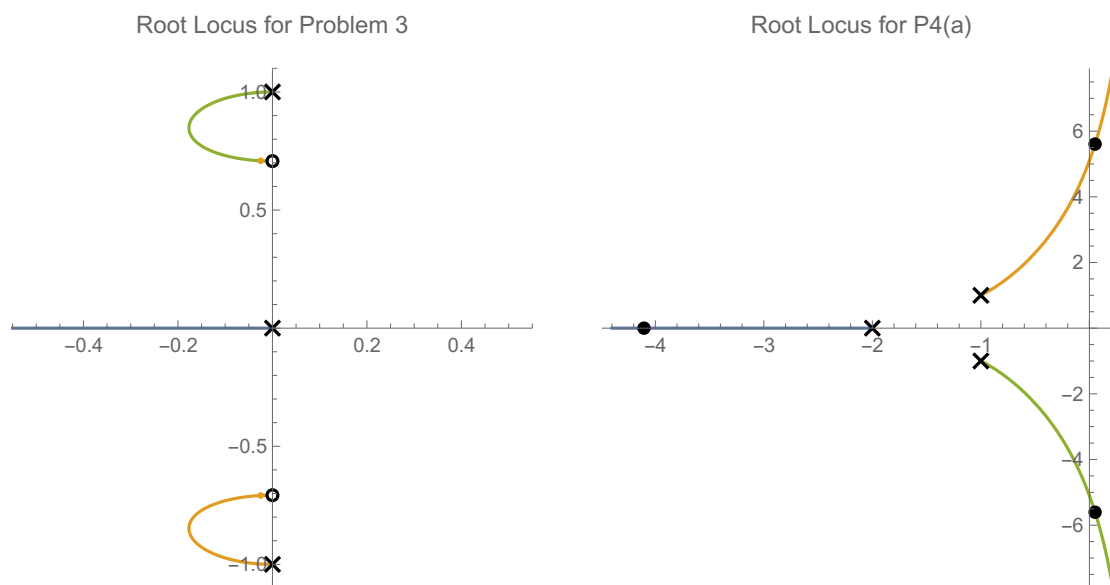

Figure 3: Left: Root locus for non-unity gain feedback and Right: All poles and zeros in LHP and still unstable.