Problem 1. Consider the following first order system:

$$\dot{y} = -0.5y + 2u, \quad y(0) = 0 \tag{1}$$

(a) (5 points) First, consider a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. As mentioned in class, it is typically important, for practical reasons, that $u(t)$ does not get too large. Consider a unit step command:

$$r(t) = \begin{cases} 0 & t < 0 \text{ sec} \\ 1 & t \geq 0 \text{ sec} \end{cases} \tag{2}$$

For what gains K_p is $|u(t)| \leq 1$ for all time? (Hint: The largest value of $|u(t)|$ will occur at $t = 0$.)

(b) (5 points) Choose the gain K_p that satisfies the constraint in part i) and minimizes the steady-state error due to the unit step command. What is the time constant of the closed-loop system for this gain?

(c) (5 points) Next consider a proportional-integral (PI) control law:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau \tag{3}$$

where $e(t) = r(t) - y(t)$ is the tracking error. Combine the system model (Equation 1) and PI controller (Equation 3) to obtain a model of the closed-loop system in the form:

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{r} + b_0 r \tag{4}$$

How do the damping ratio and natural frequency depend on K_p and K_i? What is the steady state error if r is a unit step?

(d) (10 points) Keep the value of K_p designed in part b) and choose K_i to obtain a damping ratio of $\zeta = 0.7$. For these PI gains, what are the estimated maximum overshoot and 5% settling time (neglecting the i effect of the zero)?

(e) (5 points) Plot the output response $y(t)$ due to a unit step r for both the P and PI controllers. The closed-loop with the PI controller has a zero due to the term $b_1 \dot{r}$. Briefly explain how this zero affects the response.

Solution.

(a) Note that the magnitude $|u(t)|$ of the control law is directly proportional to the difference between the state value and the reference value. i.e. $|r(t) - y(t)|$ via the gain K_p. Using the hint we see that,

$$\max_t |u(t)| = |K_p| \, |r(0) - y(0)| = |K_p| \, |1 - 0| = |K_p|$$

Therefore a preliminary condition for $|u(t)| < 1$ for all $t \in \mathbb{R}_+$ is that $|K_p| < 1$. However, note that we can write:

$$\dot{y} = -0.5y + 2u = -0.5y + 2K_p r - 2K_p y = -y(0.5 + 2K_p) + 2K_p r$$

and so when $K_p = 0$ we get a stable autonomous system with eigenvalue 1/2. On the other hand, for any constant reference signal $r(t) \equiv$ const. we have that $K_p < -0.25$ will result in an unstable system. But note, that by solving for $y(t)$ we see that further we need,

$$|u(t)| = \left| K_p \left(1 - \frac{2K_p}{0.5 + 2K_p} \right) \right| < 1$$

Therefore, the acceptable range is $K_p \in [-0.2, 1]$.

(b) For $K_p \in [-0.2, 1]$ it is clear that steady-state error to a unit-step reference input is maximized at the left end-point of the interval and minimized at the right end-point. To see why derive the transfer function,

$$\frac{Y(s)}{R(s)} = \frac{2K_p}{s + 1/2 + 2K_p}$$

and we see that for higher K_p the steady-state value approaches 1. Hence choose $K_p = 1$ to satisfy the constraint. Then

$$H(s) = \frac{2}{s + 5/2} \qquad \Longrightarrow \qquad \tau = 2/5$$

(c) Combining (1) and (3) into a single second order system results in an ODE of the form (4) with

$$a_0 = 2K_p + 1/2, \qquad a_1 = 2K_i, \qquad b_0 = 2K_i \qquad \text{and} \qquad b_1 = 2K_p$$

The characteristic polynomial then is:

$$s^2 + s(2K_p + 1/2) + 2K_i, \qquad \Longrightarrow \omega_n = \sqrt{2K_i} \quad \text{and} \quad \zeta = \frac{4K_p + 1}{4\sqrt{2K_i}}$$

For a unit-step reference signal this results in zero steady state error.

(d) Plugging in $K_p = 1$ and $\zeta = 7/10$ into the last equation above yields that $K_i = \dfrac{625}{392}$. We know,

$$M_p = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = e^{-\frac{7\pi}{\sqrt{51}}} \approxeq 4.6\%$$

and for 5% settling time,

$$T_s = \frac{-\ln(5/100)}{\zeta\omega_n} = \frac{-\ln(5/100)}{5/4} = \frac{4\ln(20)}{5} \approxeq 2.4s$$

(e) The responses are plotted in Figure 1.

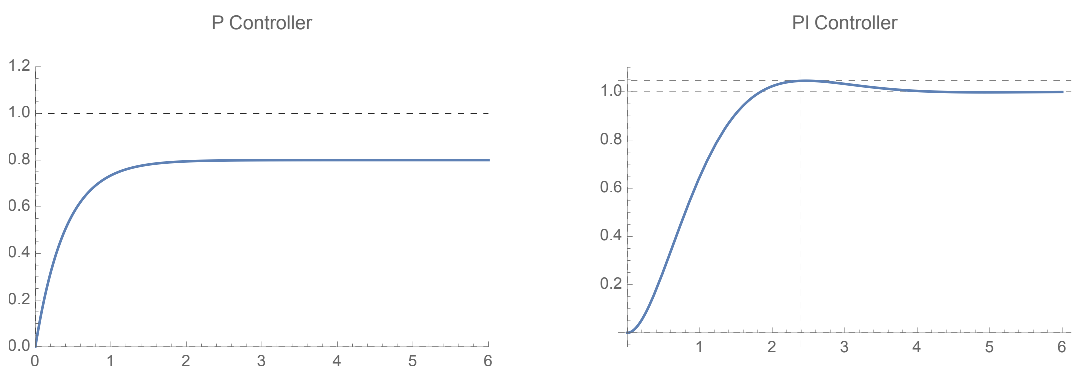

Figure 1: The response of (1) to a simple $P-$ controller and the $PI-$ controller of (3),

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The zero causes a faster response (shorter rise time) at the expense of an initial overshoot.

Problem 2. (20 points) Consider the following first order system:

$$\ddot{y} - 2\dot{y} + y = u, \qquad y(0) = 0$$

with a PD controller in the form $u_t = K_p(r(t) - y(t)) - K_d\dot{y}(t)$.

(a) What is the ODE model for the closed loop from r to y?
(b) Choose (K_p, K_d) so that the closed loop system is stable and has $(\omega_n, \zeta) = (2, 0.5)$.
(c) What is the steady state error if r is a unit step reference?
(d) Would you increase or decrease K_p to reduce the steady state error?

Solution.

(a) The required ODE is given by:

$$\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t) \tag{5}$$

(b) We can derive the transfer function as follows:

$$\begin{aligned} H(s) := \frac{Y(s)}{R(s)} &= \frac{K_p}{s^2 + (K_d - 2)s + (K_p + 1)} \\ &= \left(\frac{K_p}{K_p + 1}\right) \frac{K_p + 1}{s^2 + (K_d - 2)s + (K_p + 1)} \end{aligned}$$

So, $\omega_n^2 = K_p + 1$ and $2\zeta\omega_n = (K_d - 2)$. This means $K_p = \omega_n^2 - 1 = 3$ and $K_d = 2 \times 0.5 \times 2 + 2 = 4$.

(c) $\lim_{t\to\infty} y(t) = \lim_{s\to 0} H(s) = \dfrac{K_p}{K_p + 1}$. (FVT applies here since $H(s)$ is clearly stable by construction). This means the steady state error $\lim_{t\to\infty} r(t) - y(t) = 1 - \dfrac{K_p}{K_p + 1} = \dfrac{1}{K_p + 1}$.

(d) Clearly you would increase K_p to reduce steady state error.

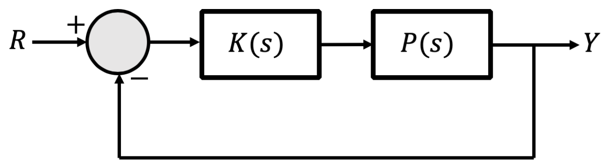

Figure 2: A diagram of a unity feedback system.

Problem 3. (20 points) Consider the unity feedback system in Figure 2. Let the plant's transfer function be given by:

$$P(s) = \frac{6.32}{s^2 - 0.12}$$

~~Suppose our controller is given by~~ $\cancel{K(s) = 4}$. Can we choose $K(s)$ as a PI controller to stabilize the closed-loop system from r to y? Apply the Routh-Hurwitz criterion to determine this.

Solution.
If our controller K is given by a PI controller, then $K(s) = K_p + \dfrac{K_i}{s}$ and the transfer function for the closed loop is given by $G(s) = \dfrac{K(s)\,P(s)}{1 + K(s)\,P(s)}$ which is:

$$G(s) = \frac{158\,(K_i + K_p s)}{25s^3 + (158K_p - 3)s + 158K_i}$$

From Routh-Hurwitz criterion for a third order polynomial $x^3 + ax^2 + bx + c$ we know that for it be stable we need $a > 0, b > 0$ and $0 < c < ab$. Since here $a = 0$ for any choice of K_p and K_i this system cannot be stabilized by a PI control.

Problem 4. Figure 3 below shows the key forces on a car. By Newton's second law, the longitudinal motion of the car is modeled by the following first-order ODE:

$$m\dot{v}(t) = F_{net}(t) - F_{aero}(t) - F_{roll} - F_{grav}(t) \tag{6}$$

where v is the velocity $(\frac{m}{sec})$, $m = 2085kg$ is the mass, and the forces are given by:

- F_{net} is the net engine force. For simplicity, assume this force is proportional to the throttle angle: $F_{net} = ku$ where $u :=$ engine throttle input (deg) and $k = 40\frac{N}{deg}$ is the force constant. The engine throttle is physically limited to remain within $0^o \leq u \leq 90^o$.
- F_{aero} is the aerodynamic drag force. For this problem we will model this as $F_{aero} = b_0 + b_1 v$ where $b_0 = -336.4N$ and $b_1 = 23.2\frac{N \cdot sec}{m}$. This approximation is accurate for velocities near $v = 29\frac{m}{sec}$.[1]
- $F_{roll} = 228N$ is the rolling resistance force due to friction at the interface of the tire and road.
- F_{grav} is the force due to gravity. This is given by $F_{grav} = mg\sin(\theta)$ where θ is the slope of the road $(rads)$ and $g = 9.81\frac{m}{sec^2}$ is the gravitational constant.

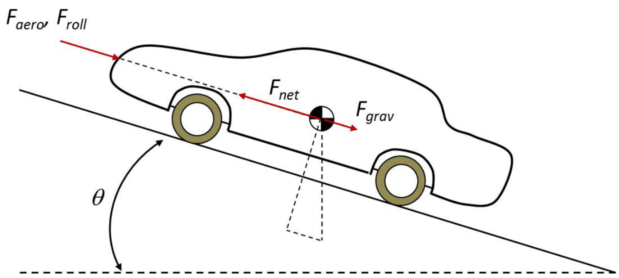

Figure 3: Free body diagram for a car.

Additional details on the model are given in Example 2.1 of the notes. Putting these pieces together yields the following first-order ODE:

$$2085\dot{v}(t) + 23.2v(t) = 40u(t) + 108.4 - F_{grav}(t) \tag{7}$$

The input is the throttle u and the output is the velocity v. The gravitational force F_{grav} is a disturbance. The homework contains a Simulink diagram `CruiseControlSim.mdl` that implements the vehicle dynamics. You can either implement the dynamics by yourself or use the provided Simulink model. For your convenience, there is also an m-file `CruiseControlPlots.m` that can be used as a template for your answers (you can also just use your own template).

[1] Additional details (not required to complete this problem): A better approximation for the aerodynamic drag is $F_{aero} = c_D v^2$ with $c_D = 0.4\frac{N \cdot sec^2}{m^2}$. This is a nonlinear function of the velocity. We can approximate this by the linear function $c_D v^2 \approx b_0 + b_1 v$. This approximation is obtained by performing a Taylor series around the velocity $\bar{v} = 29\frac{m}{sec}$.

(a) (5 points) Assume the car is on flat road so that $\theta(t) = 0rads$ and $F_{grav}(t) = 0N$. What is the open-loop (constant) input $\bar{u}$ required to maintain a desired velocity of $v_{des} = 29\frac{m}{sec}$?

(b) (5 points) Simulate the system with the input $\bar{u}$, initial condition $v(0) = 29\frac{m}{sec}$, and the following gravitational force:

$$F_{grav}(t) = \begin{cases} 0N & t < 10sec \\ 350N & t \geq 10sec \end{cases}$$

Submit a plot of velocity v versus time t. Note that the gravitational force of $350N$ corresponds to a very small road slope of $\approx 1^o$. Observe that this small slope causes a large deviation in the vehicle velocity.

(c) (10 points) Let $e(t) = v_{des} - v(t)$ denote the tracking error between the desired velocity $v_{des} = 29\frac{m}{sec}$ and actual velocity $v(t)$. Consider a PI controller of the following form:

$$u(t) = \bar{u} + K_p e(t) + K_i \int_0^t e(\tau)\, d\tau \tag{8}$$

where $\bar{u}$ is the open-loop input computed in part (a). Choose the PI gains so that the cruise control system is stable and rejects disturbances due to changing road slopes within $\approx 10sec$. The closed-loop should also be over or critically damped as oscillations are uncomfortable for the driver. *Hint:* Note that $\bar{u}$ is chosen to maintain a desired velocity $v_{des} = 29\frac{m}{sec}$ when on flat road $\theta = 0^o$. In other words, $\bar{u}$ is chosen to satisfy $23.2v_{des} = 40\bar{u} + 108.4$. Thus substituting the expression for $u(t)$ (Equation 8) into the longitudinal dynamics (Equation 7) yields:

$$2085\dot{v}(t) + 23.2v(t) = 23.2v_{des} + 40\left(K_p e(t) + K_i \int_0^t e(\tau)\, d\tau\right) - F_{grav}(t)$$

This closed-loop ODE can be used to select your gains.

(d) (10 points) Modify the Simulink diagram to include your PI controller. Simulate the closed-loop system with the your PI controller, initial condition $v(0) = 29\frac{m}{sec}$, and the following gravitational force:

$$F_{grav}(t) = \begin{cases} 0N & t < 10sec \\ 1400N & t \geq 10sec \end{cases}$$

Note that the gravitational force of $1400N$ corresponds to a road slope of $\approx 4^o$. You will need to update the Simulink block that generates this gravitational force.
Submit plots of velocity v versus time t and throttle input u versus t. Verify that the throttle input remains within the physical limits. You should also submit the Simulink diagram modified to include your PI controller.

Solution.

(a) We have that

$$23.2v_{des} = 40\bar{u} + 108.4 \implies \bar{u} = 14.11.$$

(b) See Figure 4 for the plot, and attached Simulink file `hw3_1b.slx` for the Simulink diagram.

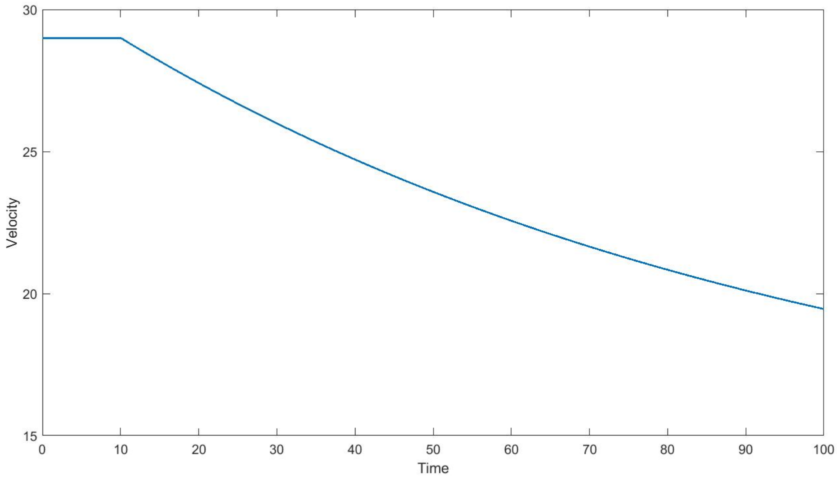

Figure 4: 1(b) Velocity vs Time plot.

(c) Now,

$$e(t) = v_{des} - v(t) \implies \dot{e}(t) = -\dot{v}(t), \ddot{e}(t) = -\ddot{v}(t).$$

Substituting this in the equation given in the hint, we get

$$-2085\dot{e}(t) - 23.2e(t) = 40\left(K_P e(t) + K_i \int_0^t e(\tau)d\tau\right) - F_{grav}(t),$$

rearranging which we get the closed loop error dynamics as

$$2085\dot{e}(t) + (23.2 + 40K_p)e(t) + 40K_i \int_0^t e(\tau)d\tau = F_{grav}(t).$$

We assume that the gravitational force acts as a step input of magnitude F_g here. This is done purely to make the analysis of the system easier, and will be close to reality if the slopes on the road are more or less constant for large enough intervals of time. Taking Laplace transforms, we can then say that

$$E(s) = \frac{F_g}{2085s^2 + (23.2 + 40K_p)s + 40K_i},$$

which means the error is the impulse response of a second order system. There are two requirements given in the question, one of which is that the system needs to be overdamped. This is easy to ensure now that we have the closed loop response in the form of a second order system response.

The other requirement was to ensure that the disturbances are rejected within 10s, whose precise description was for you to figure out. One way to do this is to ensure that the 95% settling time of the error dynamics is less than 10s, which means the effects of a changing road slope (modeled as a step input into the error dynamics) is reduced to within 5% of its initial value in 10s. This is a reasonable model of what it means to "reject the disturbances" in 10s. We see that

$$2\zeta\omega_n = \frac{23.2 + 40K_p}{2085}, \quad \omega_n^2 = \frac{40K_i}{2085}.$$

The 95% settling time is approximately $\dfrac{3}{\zeta\omega_n}$. A subtle point to note is that the 95% settling time as given above was computed for the step response of a second order system, while the error here is the impulse response of a second order system. However, a similar analysis holds in this case as well giving the same formula. We need to ensure

$$\frac{6.6\zeta - 1.6}{\omega_n} \leq 10, \zeta \geq 1.$$

Here I have used the approximation formula for settling time when $\zeta > 0.7$ in the first condition. Let us fix $\zeta = 1.1$, which gives us

$$\frac{5.66}{\omega_n} \leq 10 \implies \omega_n \geq 0.566$$

Let us choose $\omega_n = 0.566$ which gives

$$K_i = \frac{2085}{40} \times 0.566 \times 0.566 = 16.68.$$

We can also find K_p as

$$2 \times 1.1 \times 0.566 = \frac{23.2 + 40K_p}{2085} \implies K_p = \frac{1.25 \times 2085 - 23.2}{40} = 64.57.$$

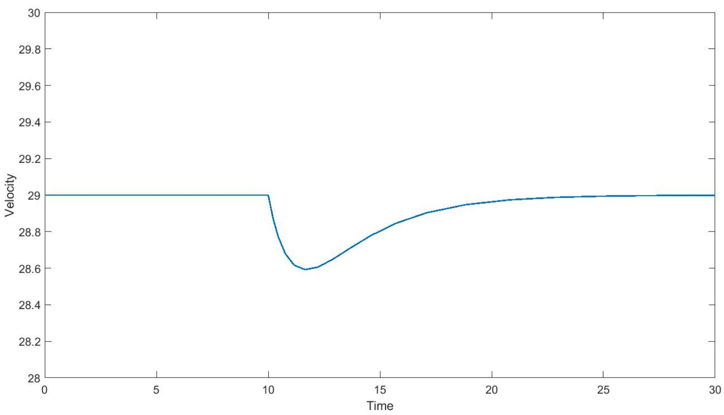

Figure 5: 1(d) Velocity vs Time plot.

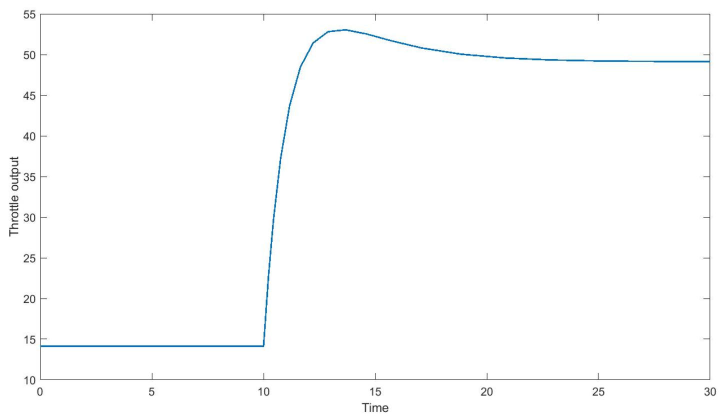

Figure 6: 1(d) Throttle vs Time plot.

(d) See Figure 5 and Figure 6 for the plots. I used $K_p = 64.57$ and $K_i = 16.68$. The requirements are clearly met reasonably.