Problem 1. Consider the dynamics for the mass-spring system, as depicted in Figure 1. The dynamics are governed by the following state-space model:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\dfrac{k}{m} & -\dfrac{\rho}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \dfrac{1}{m}\end{bmatrix} u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Here, k is the spring constant and ρ is the friction coefficient (yes, the mass m in the figure does not touch the floor, but *assume it does!*).

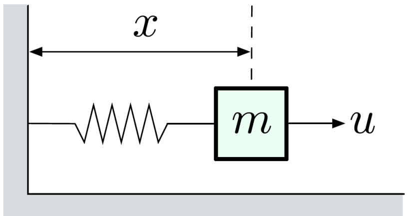


Figure 1: The mass-spring system.

(a) (5 points) Find the transfer function of this system from u to y.

(b) (15 points) Suppose that the C matrix is replaced, such that:

$$y = \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Recalculate the transfer function with this sensor model. Write ω_n and ζ in terms of k, ρ, m. Draw a block diagram for this transfer function using integrator, summation, and gain blocks.

Solution.

Recall that for a general state-space model:

$$\dot{x} = Ax + Bu$$
$$y = Cx + D$$

the transfer function from u to y is: $G(s) = C(sI - A)^{-1}B + D$.

(a) We have $A = \begin{bmatrix}0 & 1\\ -\dfrac{k}{m} & -\dfrac{\rho}{m}\end{bmatrix}$, $B = \begin{bmatrix}0\\ \dfrac{1}{m}\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\end{bmatrix}$, and $D = [\,]$. Therefore, we have:

$$\begin{aligned}G_1(s) &= C(sI - A)^{-1}B + D\\ &= \begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -\dfrac{k}{m} & -\dfrac{\rho}{m}\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\ \dfrac{1}{m}\end{bmatrix}\\ &= \frac{1}{ms^2 + \rho s + k}.\end{aligned}$$

(b) In this case, we have the same A and B matrices but with different D matrix. Therefore, we have:

$$\begin{aligned} G_2(s) &= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &= \frac{c_1 + c_2 s}{ms^2 + \rho s + k} = \frac{c_1/m + (c_2/m)s}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \end{aligned}$$

where $2\zeta\omega_n = \frac{\rho}{m}$, $\omega_n^2 = \frac{k}{m}$. Hence we have:

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{\rho}{2\sqrt{km}}.$$

The block diagram is shown as follows:

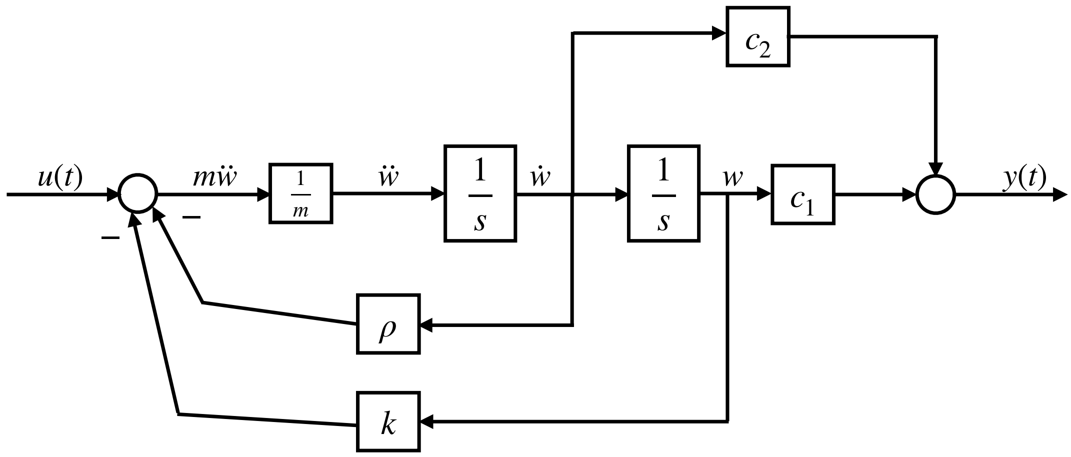


Figure 2: The block diagram for Problem 1.

Problem 2. Consider the transfer function:

$$H(S) = \frac{25}{s^2 + 6s + 25}$$

(a) (5 points) Draw a block diagram for $H(s)$ using integrator, summation, and gain blocks.

(b) (5 points) Suppose you are given the following time-domain specs: rise time $t_r \le 0.6$ and settling time $t_s \le 1.6$. (Here we're considering settling time to within 5% of the steady-state value.) Plot the admissible pole locations in the s-plane corresponding to these two specs. Does this system satisfy these specs?

(c) (5 points) Repeat the previous problem for the specs: rise time $t_r \le 0.6$, settling time $t_s \le 1.6$, and magnitude $M_p \le 1/e^2$. Plot the admissible pole locations; does this system satisfy these specs?

(d) (5 points) Draw a block diagram for $(s + 1)H(s)$ using integrator, summation, and gain blocks.

Solution. (a) The block diagram can be found in Figure 3.

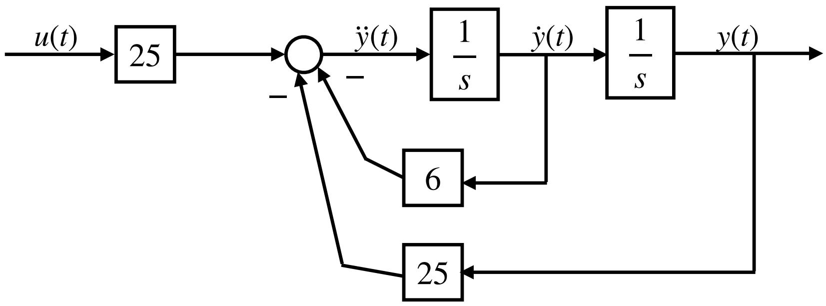


Figure 3: The block diagram for Problem 2(a).

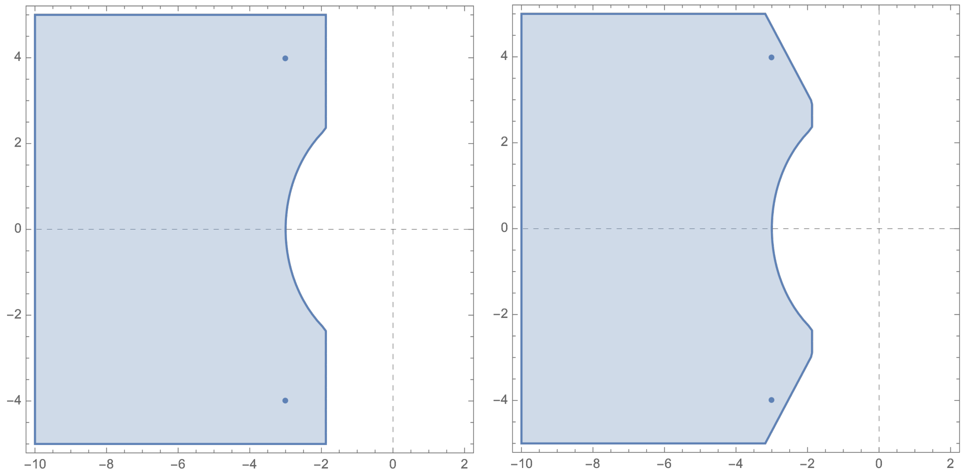


Figure 4: Admissible pole locations for 2(b) and 2(c).

(b) Consider a second order system:

$$H(s) = \frac{b_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

If $\zeta < 1$, we have two poles: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. We want $t_r \leq 0.6$ and $t_s \leq 1.6$, which is equivalent to say:

$$\frac{1.8}{\omega_n} < 0.6, \ \frac{3}{\zeta\omega_n} < 1.6 \Longrightarrow |s| \geq 3, \ \mathrm{Re}\{s\} < -\frac{15}{8}.$$

Therefore, the admissible pole locations in the s-plane can be found in Figure 4 (left plot). For this specific system, we have $2\zeta\omega = 6$, $\omega_n^2 = 25$, we have

$$\frac{1.8}{\omega_n} = 0.36 < 0.6, \quad \frac{3}{\zeta\omega_n} = 1 < 1.6.$$

The poles fall into the admissible region, so this system satisfies these specs.

(c) For the magnitude, we have:

$$M_p \leq \frac{1}{e^2} \Longrightarrow e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \leq e^{-2} \Longrightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} \geq \frac{2}{\pi} \Longrightarrow \frac{|\mathrm{Re}\{s\}|}{|\mathrm{Im}\{s\}|} \geq \frac{2}{\pi}$$

Combining the conditions obtained in (b), the admissible pole locations in the s-plane can be found in Figure 4 (right plot). For this specific system, we have

$$\frac{|\mathrm{Re}\{s\}|}{|\mathrm{Im}\{s\}|} = \frac{3}{4} \geq \frac{2}{\pi}.$$

The poles fall into the admissible region, so the system satisfies the specs.

(d) The block diagram can be found in Figure 5.

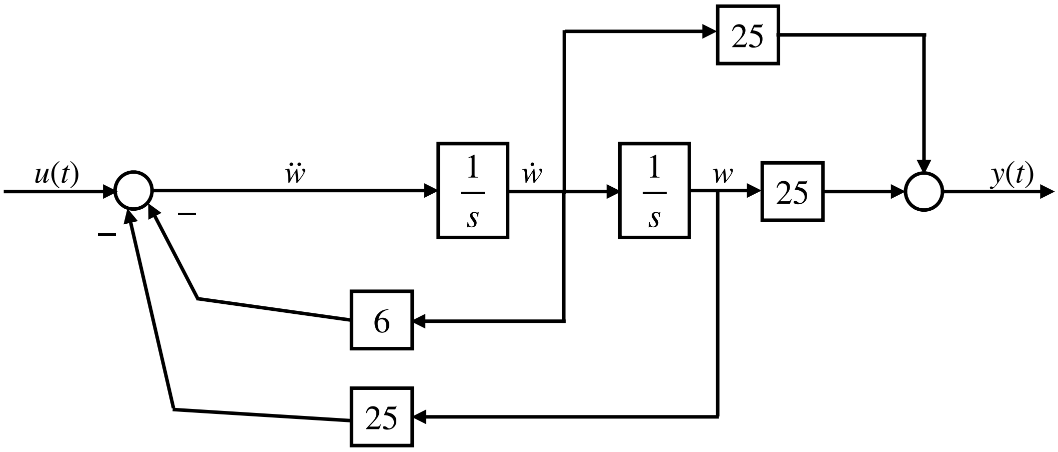


Figure 5: The block diagram for Problem 2(d).

Problem 3. (15 points) Consider the unity feedback system in Figure 6. Let the plant's transfer function be given by:

$$P(s) = \frac{1}{s^3 + 2s^2 + 3s + 1}$$

Suppose our controller is given by $K(s) = 4$. What is the transfer function from R to Y? How to convert that transfer function to a linear state-space model? Use the Routh-Hurwitz criterion to determine whether this model is stable or not.

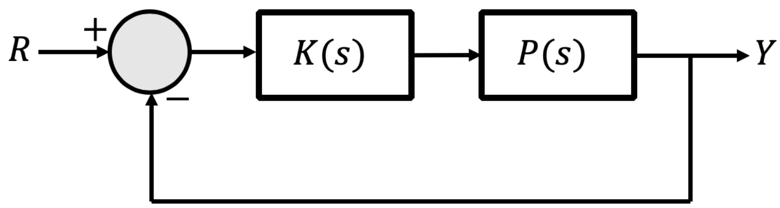


Figure 6: A diagram of a unity feedback system.

Solution.

The transfer function from R to Y is given by $G(s) = \dfrac{K(s)P(s)}{1 + K(s)P(s)}$ which is

$$G(s) = \frac{4}{s^3 + 2s^2 + 3s + 5}$$

Recall that one state space model (called the controllable canonical form) for a system with closed loop transfer function:

$$\frac{N(s)}{D(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T$$

and

$$C = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & b_{n-2} - a_{n-2} b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \qquad D = b_0$$

Therefore for us we have,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \text{ and } \quad D = 0$$

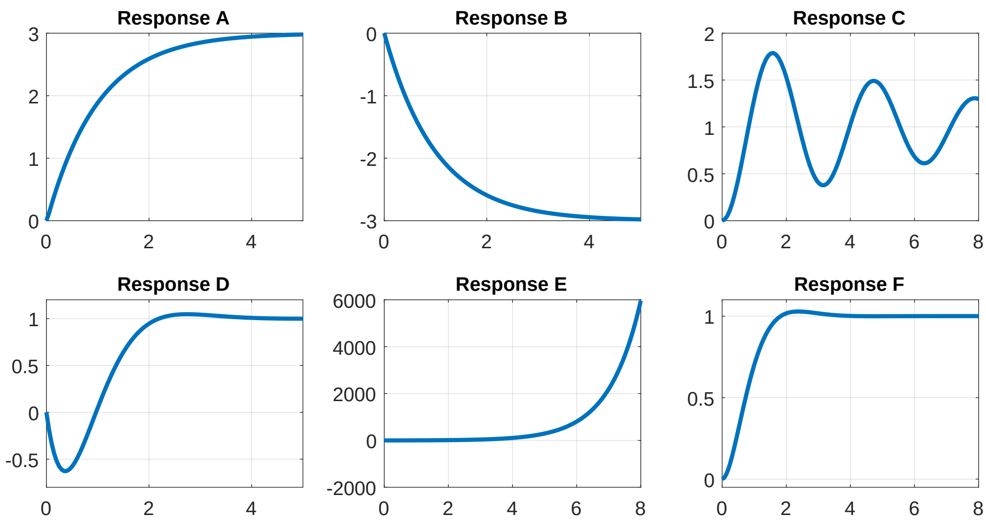
The model is stable because we the table below and there are no sign changes in the first row.

s^3	s^2	s^1	s^0
1	2	0.5	5
3	5	0	0

Problem 4. (30 points) Consider the six transfer functions given below. For each $G_i(s), i = 1, 2, \dots 6$, specify the following **in turn**: (a) poles, (b) zeros (if any), (c) stable or unstable, and (d) steady-state gain **before proceeding to the next**. Use these answers to match each of the six transfer functions with one of the unit step responses in the figure below. All responses were generated with zero initial conditions.

$$G_1(s) = \frac{-4s+4}{s^2+3s+4} \qquad G_2(s) = \frac{4}{s^2+0.3s+4} \qquad G_3(s) = \frac{4}{s^2+3s+4}$$

$$G_4(s) = \frac{-s+3}{s-1} \qquad G_5(s) = \frac{300}{s^2+101s+100} \qquad G_6(s) = \frac{-3}{s+1}$$



Solution.
The solution is as per Table 1

	(a) Poles	(b) Zeros	(c) Stable?	(d) Steady-state gain	Response
$G_1(s)$	$-3/2 \pm i\sqrt{7}/2$	1	Yes	1	D
$G_2(s)$	$\frac{1}{20}\left(-3 \pm i\sqrt{1591}\right)$	None	Yes	1	C
$G_3(s)$	$-3/2 \pm i\sqrt{7}/2$	None	Yes	1	F
$G_4(s)$	1	3	No	N/A	E
$G_5(s)$	-100 and -1	None	Yes	3	A
$G_6(s)$	-1	None	Yes	-3	B

Table 1: Table of results for Problem 4

The only unstable system is $G_4(s)$ and it matches with the unstable response E. The steady state gains of $G_5(s)$ and $G_6(s)$ make it clear they match to response A and B respectively. Between $G_2(s)$ and $G_3(s)$ they have the same natural frequencies but vastly different damping ratios; absent any zeros this makes it clear that the lightly damped $G_2(s)$ corresponds to response C and the more damped $G_3(s)$ corresponds to response F. Finally note $G_1(s) = G_3(s) - sG_3(s)$ so that the step response of $G_1(s)$ will the step response of $G_1(s)$ less its derivative.

Problem 5. (15 points) Without a computer, determine whether or not the following polynomials have any RHP roots:

(a) $s^6 + 2s^5 + 3s^4 + s^3 + s^2 - 3s + 5$
(b) $s^4 + 10s^3 + 10s^2 + 20s + 1$
(c) $s^4 + 10s^3 + 10s^2 + 1$

Solution. (a) This system is unstable,

s^6	s^5	s^4	s^3	s^2	s^1	s^0
1	2	5/2	-1	-15	-22/3	5
3	1	5/2	-7	5	0	0
1	-3	5	0	0	0	0
5	0	0	0	0	0	0

Table 2: Routh table for part (a) - note the table is transposed

(b) This system is stable.
(c) This system is unstable.

s^4	s^3	s^2	s^1	s^0
1	10	8	75/4	1
10	20	1	0	0
1	0	0	0	0

s^4	s^3	s^2	s^1	s^0
1	10	10	-1	1
10	0	1	0	0
1	0	0	0	0

Table 3: Routh tables for part (b) and (c) - note the tables are transposed