Problem 1. Calculate the following Laplace transforms $F_i = \mathcal{L}\{f_i\}$ **by hand**:

(a) $f_1(t) = 3\cos(t) + 2\sin(t)$

Hint: Recall Euler's formula.

(b) $f_2(t) = e^{-3t}$

(c) $f_3(t) = 3\cos(t) + 2\sin(t) + e^{-3t}$

Solution.

Recall that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ for all $a \in \mathbb{C}$, and $e^{it} = \cos(t) + i\sin(t)$ (Euler's formula). Therefore,

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \text{and} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

In addition, we have

$$\mathcal{L}\{\cos(t)\} = \mathcal{L}\left\{\frac{e^{it} + e^{-it}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{s}{s^2+1},$$
$$\mathcal{L}\{\sin(t)\} = \mathcal{L}\left\{\frac{e^{it} - e^{-it}}{2i}\right\} = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{s^2+1}.$$

(a) So then,

$$\mathcal{L}\{3\cos(t) + 2\sin(t)\} = 3\mathcal{L}\{\cos(t)\} + 2\mathcal{L}\{\sin(t)\} = \frac{3s+2}{s^2+1}$$

(b) and similarly,

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$$

(c) and finally,

$$\mathcal{L}\{3\cos(t) + 2\sin(t) + e^{-3t}\} = \frac{3s+2}{s^2+1} + \frac{1}{s+3} = \frac{4s^2 + 11s + 7}{(s^2+1)(s+3)}.$$

Problem 2. Consider the following four transfer functions.

$$H_1(s) = \frac{4}{s+10} \qquad H_2(s) = \frac{4}{s-10} \qquad H_3(s) = \frac{4}{s^2+13s+30} \qquad H_4(s) = \frac{4}{s^2+2s+2}$$

For each part below, answer the questions for $H_i, i = 1, 2, 3, 4$ in turn before proceeding to next part.

(a) What are the poles and zeros?
(b) What is the general form of the free response?
(c) What is the general form of the forced response?
(d) Recall that a step response is the output of the system when all the initial conditions are zero and input is as follows:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Compute the step responses **by hand**. Calculate the steady-state value of each step response.

(e) Next, calculate the response with zero initial conditions and the following input (5 points):

$$u(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(f) Finally, calculate the response with zero initial conditions and the following input (5 points):

$$u(t) = \begin{cases} t^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Solution. (a) None of them have zeros. For poles, we answer each transfer function in turn:

$H_1(s)$: $s + 10 = 0 \Rightarrow s = -10$

$H_2(s)$: $s - 10 = 0 \Rightarrow s = 10$

$H_3(s)$: $s^2 + 13s + 30 = 0 \Rightarrow s = -3, -10$

$H_4(s)$: $s^2 + 2s + 2 = 0 \Rightarrow s = -1 \pm i$

(b) For the general form of the free response, we answer each transfer function in turn:

$H_1(s)$: $y(t) = ce^{-10t}$

$H_2(s)$: $y(t) = ce^{10t}$

$H_3(s)$: $y(t) = c_1e^{-3t} + c_2e^{-10t}$

$H_4(s)$: $y(t) = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$

We can obtain the solution coefficients from initial conditions of ODEs. For example, for $H_3(s)$, we have:

$$\begin{cases} c_1 + c_2 = y(0) \\ -3c_1 - 10c_2 = y'(0) \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{10y(0)+y'(0)}{7} \\ c_2 = -\frac{3y(0)+y'(0)}{7} \end{cases}$$

(c) For the general form of the forced response, we answer each transfer function in turn:

$H_1(s)$: $y(t) = y_p(t) + ce^{-10t}$

$H_2(s)$: $y(t) = y_p(t) + ce^{10t}$

$H_3(s)$: $y(t) = y_p(t) + c_1e^{-3t} + c_2e^{-10t}$

$H_4(s)$: $y(t) = y_p(t) + c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$

where $y_p(t)$ is the particular solutions to the homogeneous ODEs. Similar to part (b), we can obtain solution coefficients c's from initial conditions.

(d) We answer each transfer function in turn:

$H_1(s)$: Let $y_p(t) = c_0$, which is a constant. Substituting this into the ODE, we have:

$$10c_0 = 4 \Longrightarrow c_0 = 0.4.$$

Then, we have $y(t) = 0.4 + ce^{-10t}$. Since $y(0) = 0$, we have $0.4 + c = 0 \Rightarrow c = -0.4$. Therefore, the step response of $H_1(s)$ with zero initial condition is:

$$y(t) = 0.4 - 0.4e^{-10t}.$$

The steady-state value is $y(\infty) = 0.4$.

$H_2(s)$: Same as $H_1(s)$, we have $y(t) = -0.4 + 0.4e^{10t}$. The steady-state value is $y(\infty) = +\infty$.

$H_3(s)$: Let $y_p = c_0$. Substituting this into the ODE, we have:

$$30c_0 = 4 \Longrightarrow c_0 = \frac{2}{15}.$$

Then we have $y(t) = \frac{2}{15} + c_1e^{-3t} + c_2e^{-10t}$. To obtain c_1 and c_2, applying $y(0) = y'(0) = 0$:

$$\begin{cases} \frac{2}{15} + c_1 + c_2 = 0 \\ -3c_1 - 10c_2 = 0 \end{cases} \longrightarrow \begin{cases} c_1 = -\frac{4}{21} \\ c_2 = \frac{2}{35} \end{cases}$$

Therefore, the step response of $H_3(s)$ with zero initial condition is:

$$y(t) = \frac{2}{15} - \frac{4}{21}e^{-3t} + \frac{2}{35}e^{-10t}.$$

The steady-state value is $y(\infty) = \frac{2}{15}$.

$H_4(s)$: Same as $H_3(3)$, we have $y(t) = 2 - 2e^{-t}\cos(t) - 2e^{-t}\sin(t)$. The steady-state value is $y(\infty) = 2$.

(e) We answer each transfer function in turn:

$H_1(s)$: Let $y_p = k_0 + k_1 t$. Substituting this into the ODE, we have:

$$k_1 + 10k_0 + 10k_1 t = 4t \Longrightarrow \begin{cases} k_1 + 10k_0 = 0 \\ 10k_1 = 4 \end{cases} \Longrightarrow \begin{cases} k_0 = -0.04 \\ k_1 = 0.4 \end{cases}$$

Then we have $y(t) = -0.04 + 0.4t + ce^{-10t}$. To obtain c, applying $y(0) = 0$:

$$-0.04 + c = 0 \longrightarrow c = 0.04.$$

Therefore, the ramp response of $H_1(s)$ with zero initial condition is:

$$y(t) = -0.04 + 0.4t + 0.04e^{-10t}.$$

$H_2(s)$: Similar to $H_1(s)$, we have:

$$y(t) = -0.04 - 0.4t + 0.04e^{10t}.$$

$H_3(s)$: Let $y_p = k_0 + k_1 t$. Substituting this into the ODE, we have:

$$13k_1 + 30k_0 + 30k_1 t = 4t \Longrightarrow \begin{cases} 13k_1 + 30k_0 = 0 \\ 30k_1 = 4 \end{cases} \Longrightarrow \begin{cases} k_0 = -\dfrac{13}{225} \\ k_1 = \dfrac{2}{15} \end{cases}$$

Then we have $y(t) = -\dfrac{13}{225} + \dfrac{2}{15}t + c_1 e^{-3t} + c_2 e^{-10t}$. To obtain c_1 and c_2, applying $y(0) = y'(0) = 0$:

$$\begin{cases} c_1 + c_2 - \dfrac{13}{225} = 0 \\ 3c_1 + 10c_2 - \dfrac{2}{15} = 0 \end{cases} \longrightarrow \begin{cases} c_1 = \dfrac{4}{63} \\ c_2 = -\dfrac{1}{175} \end{cases}$$

Therefore, the ramp response of $H_2(s)$ with zero initial condition is:

$$y(t) = -\frac{13}{225} + \frac{2}{15}t + \frac{4}{63}e^{-3t} - \frac{1}{175}e^{-10t}.$$

$H_4(s)$: Similar to $H_3(s)$, we have:

$$y(t) = -2 + 2t + 2e^{-t}\cos(t).$$

(f) We answer each transfer function in turn:

$H_1(s)$: Let $y_p = k_0 + k_1 t + k_2 t^2$. Substituting this into the ODE, we have:

$$k_1 + 2k_2 t + 10k_0 + 10k_1 t + 10k_2 t^2 = 4t^2 \Longrightarrow \begin{cases} k_0 = 0.008 \\ k_1 = -0.08 \\ k_2 = 0.4 \end{cases}$$

Then we have $y(t) = 0.008 - 0.08t + 0.4t^2 + ce^{-10t}$. To obtain c, applying $y(0) = 0$:

$$0.008 + c = 0 \longrightarrow c = -0.008$$

Therefore, the quadratic response of $H_1(s)$ with zero initial condition is:

$$y(t) = 0.008 - 0.08t + 0.4t^2 - 0.008e^{-10t}.$$

$H_2(s)$: Similar to $H_1(s)$, we have:

$$y(t) = -0.008 - 0.08t - 0.4t^2 + 0.008e^{10t}.$$

$H_3(s)$: Let $y_p = k_0 + k_1 t + k_2 t^2$. Substituting this into the ODE, we have:

$$2k_2 + 13(k_1 + 2k_2 t) + 30(k_0 + k_1 t + k_2 t^2) = 4t^2 \implies \begin{cases} k_0 = \frac{139}{3375} \\ k_1 = -\frac{26}{225} \\ k_2 = \frac{2}{15} \end{cases}$$

Then we have $y(t) = \frac{139}{3375} - \frac{26}{225}t + \frac{2}{15}t^2 + c_1 e^{-3t} + c_2 e^{-10t}$. To obtain c_1 and c_2, applying $y(0) = y'(0) = 0$:

$$\begin{cases} c_1 + c_2 + \frac{139}{3375} = 0 \\ 3c_1 + 10c_2 + \frac{26}{225} = 0 \end{cases} \longrightarrow \begin{cases} c_1 = -\frac{8}{189} \\ c_2 = \frac{1}{875} \end{cases}$$

Therefore, the quadratic response of $H_2(s)$ with zero initial condition is:

$$y(t) = \frac{139}{3375} - \frac{26}{225}t + \frac{2}{15}t^2 - \frac{8}{189}e^{-3t} + \frac{1}{875}e^{-10t}.$$

$H_4(s)$: Similar to $H_3(s)$, we have:

$$y(t) = 2 - 4t + 2t^2 - 2e^{-t}\cos(t) + 2e^{-t}\sin(t).$$

Problem 3. Consider the following transfer functions:

(a) $H_1(s) = \frac{1}{s^2 - s + 3}$

(b) $H_2(s) = \frac{s-3}{s^2 + 5s + 4}$

(c) $H_3(s) = \frac{3s+7}{s^2 + 8s + 7}$

(d) $H_4(s) = \frac{5}{s^2 + 6s + 25}$

Calculate $\lim_{s\to 0} H_i(s)$. Use the MATLAB command `step` to plot their step responses, and attach the plots (the command `ltiview` may also be useful, be sure to select plot pages during upload to Gradescope). Can the Final Value Theorem be invoked? What is the DC gain?

Solution. (a) We have that $\lim_{s\to 0} H_1(s) = 1/3$. The poles are at $1/2 \pm i\sqrt{11}/2$ and therefore FVT cannot be invoked and the DC gain is not defined as seen from the plot of Figure 1(a).

(b) Now $\lim_{s\to 0} H_2(s) = -3/4$. See Figure 1(b) for the step response. The FVT does apply here since both poles $(-4, -1)$ lie in the LHP. The DC gain is -0.75, which can be seen both in the plot and by FVT.

(c) The $\lim_{s\to 0} H_3(s) = 1$. See Figure 1(c) for the step response. The FVT does apply here since both poles $(-7, -1)$ lie in the LHP. The DC gain is 1, which can be seen both in the plot and by FVT.

(d) Finally $\lim_{s\to 0} H_4(s) = 1/5$. See Figure 1(d) for the step response. The FVT does apply here since both poles $(-3 \pm \sqrt{4}i)$ lie in the LHP. The DC gain is 0.2, which can be seen both in the plot and by FVT.

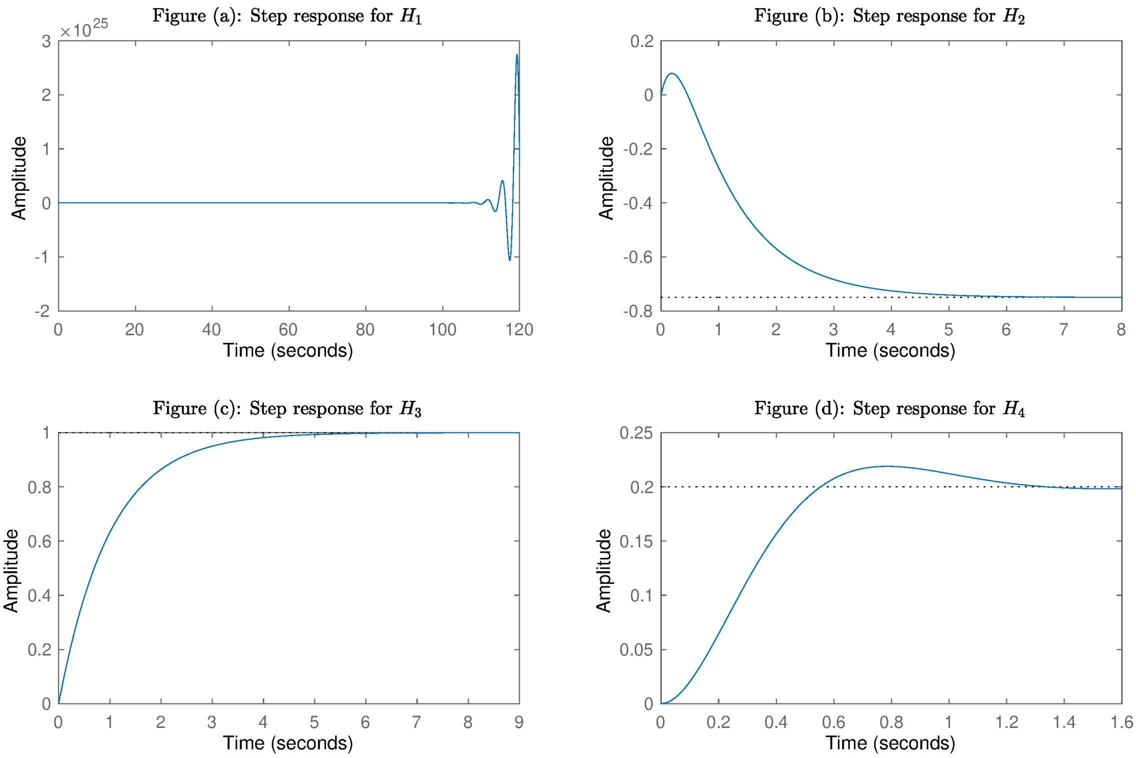


Figure 1: Step responses for the four transfer functions.

The code to generate the plots is shown below:

```
sys1 = tf(1, [1 -1 3]);
sys2 = tf([1, -3], [1, 5, 4]);
sys3 = tf([1, 7], [1, 8 , 7]);
sys4 = tf(5, [1, 6, 25]);

systems = {sys1, sys2, sys3, sys4};
labels = {"(a)", "(b)", "(c)", "(d)"};

f = figure;
for i=1:4
    subplot(2,2,i)
    step(systems{i})
    figtitle = sprintf("Figure %s:  Step response for $H_{%d}$",labels{i}, i);
    title(figtitle,'Interpreter','latex')
end
```

Problem 4. Sketch the response $y(t)$ vs. t for the system, initial conditions, and input given below. Label the steady-state value of y and the approximate settling time. Also label the approximate peak value of y. Specify whether the system is over or underdamped.

$$\ddot{y} + 6\dot{y} + 25y = 25u, \tag{1}$$

$$y(0) = 0,\ \dot{y}(0) = 0,\ u(t) = \begin{cases} 0 & t < 0 \text{ sec} \\ 4 & t \geq 0 \text{ sec} \end{cases} \tag{2}$$

Finally, suppose you are given the following time-domain specs: rise time $t_r \leq 0.6$ and settling time $t_s \leq 1.6$. (Here we're considering settling time to within 5% of the steady-state value.) Plot the admissible pole locations in the s-plane corresponding to these two specs. Does this system satisfy these specs?

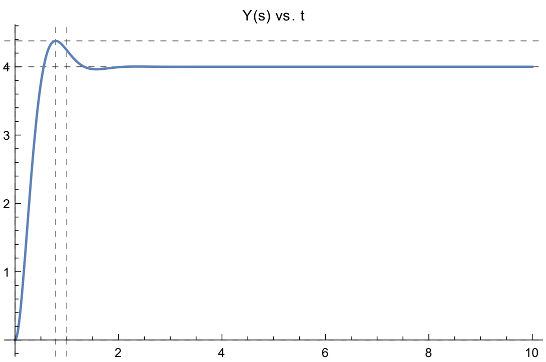
Solution.
Laplace transforming the given differential equation, control signal and input we get:

$$Y(s) = \frac{4}{s} - \frac{4(6+s)}{s^2+6s+25}$$

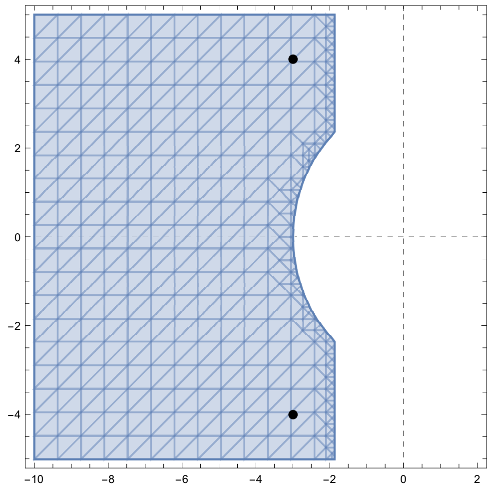
which on Inverse Laplace Transforming yields:

$$y(t) = 4 - e^{-3t}\left(4\cos(4t) + 3\sin(4t)\right)$$

which makes it obvious that the steady-state value is 4. Note that we could have obtained this by looking at $\lim_{s\to 0} sY(s)$. The sketch of the response is given below with the approximate peak value and settling times marked. To do so we write $Y(s) = \frac{s}{4} \cdot H(s)$ where



(a) Sketch of the output response with $t_s \approx 1s$, $y_p = 4.38$ and steady-state value of $y_{ss} = 4$.



(b) Shaded region is the admissible pole locations and the two black dots are poles showing that the constraints are satisfied.

Figure 2: The two plots and sketches for Problem 4

$$H(s) := \frac{25}{s^2 + 6s^2 + 25}$$

allowing us to represent the given system as a standard second order system response to four times the unit step function. This gives

$$\omega_n = 5, \qquad \zeta = 3/5, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 4, \qquad \text{and} \qquad \sigma = \zeta\omega_n = 3$$

which signifies an under-damped system. Then we have that

$$t_s \approx \frac{3}{\sigma} = 1s, \qquad \text{and} \qquad y_p(t) = (1 + M_p)\cdot y_{ss}(t) \qquad \text{where} \qquad M_p := \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

as shown in Figure 2(a). Now Given that $t_r = \frac{1.8}{\omega_n}$ and $t_s = \frac{3}{\sigma}$ for the specification that $t_r \leq 6/10$ and $t_s \leq 16/10$ we conclude that the requirements translate to $\omega_n \geq 3$ and $\sigma > 15/8$. The admissible pole location is the shaded region in the Figure 2(b) below and the poles of the system, $-3 \pm 4i$ are the two black dots $\implies$ constraints are satisfied.