Problem 1. Consider a linear, time-invariant system with the following impulse response:

$$\dot{y} = -5y + 2r$$

where r is the reference signal.

(a) (5 points) Compute the transfer function from R to Y. What are the poles, zeros, and DC gain?
(b) (3 points) Draw a block diagram for this system using integrator, summation, and gain blocks.
(c) (2 points) What is the general form of the free response?
(d) (2 points) What is the general form of the forced response?
(e) (4 points) Assuming $y(0) = 0$. Compute the step response by hand.
(f) (4 points) Now suppose that the reference signal is the ramp: $r(t) = t$ for $t \geq 0$. Assuming $y(0) = 0$. Compute the response $y(t)$ by hand.

Solution.

(a) From the ODE, we have $sY(s) + 5Y(s) = 2R(s)$, therefore, the transfer function from R to Y is:
$$H(s) = \frac{Y(s)}{R(s)} = \frac{2}{s+5}.$$
The pole of the above transfer function is $s = -5$, there is no zero, and the DC gain is $H(0) = \frac{2}{5}$.
(b) The block diagram for this system can be found in Figure 1.
(c) Since we have one pole $s = -5$, the general form of the free response is: $y(t) = Ce^{-5t}$, where C can be determined by the initial condition $y(0)$.
(d) The general form of the forced response is $y(t) = y_p(t) + Ce^{-5t}$, where $y_p(t)$ is a particular solution of the above ODE. C can be determined by the initial condition $y(0)$.
(e) Assume that $y_p(t) = C_0$, which is a constant. Then we have $\dot{y}_p(t) = 0$. Substituting this into the ODE leads to $0 = -5C_0 + 2$, which gives $C_0 = \frac{2}{5}$. Therefore, we have $y(t) = \frac{2}{5} + Ce^{-5t}$. Since $y(0) = 0$,
$$\frac{2}{5} + Ce^0 = \frac{2}{5} + C = 0 \Longrightarrow C = -\frac{2}{5}.$$
Hence $y(t) = \frac{2}{5} - \frac{2}{5}e^{-5t}$.
(f) Now assume that $y_p(t) = k_0 + k_1 t$, $\dot{y}_p(t) = k_1$. Substituting this into the ODE gives
$$k_1 = -5k_0 - 5k_1 t + 2t \implies \begin{cases} k_0 = -\frac{2}{25} \\ k_1 = \frac{2}{5} \end{cases}$$
Hence we have $y(t) = -\frac{2}{25} + \frac{2}{5}t + Ce^{-5t}$. Since $y(0) = 0$:
$$-\frac{2}{25} + Ce^0 = 0 \Longrightarrow C = \frac{2}{25}.$$
Hence $y(t) = -\frac{2}{25} + \frac{2}{5}t + \frac{2}{25}e^{-5t}$.

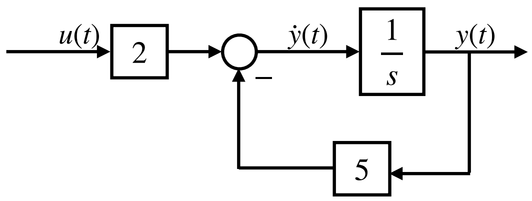


Figure 1: Block diagram of the system

Problem 2.

(a) (5 points) Consider the following second-order system.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For the step response, suppose we have the following specifications:

- rise time $t_r \leq 1.8$
- settling time $t_s \leq 6.5$

Plot the pole locations that achieve this on the axes provided below. **Clearly label your axes.** What conditions must be satisfied by the ω_n and ζ parameters to meet these specifications? *Be sure to state your conditions in terms of* ω_n *and* ζ*!*

(b) (5 points) Consider the following second-order system.

$$H(s) = \frac{3}{s^2 + 4s + 3}$$

Does the step response of this system satisfy the following specifications?

- rise time $t_r \leq 1.8$
- settling time $t_s \leq 6.5$

(c) For the second-order system in (b). Now consider

$$y(0) = 1 \qquad \dot{y}(0) = 2$$

$$u(t) = \begin{cases} 2 - 2e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Apply Final Value Theorem (FVT) to calculate the steady state value of y. Provide detailed explanations justifying how FVT is applied in this case. (You are allowed to verify your answers using `Matlab` but you need to first provide all the derivations using FVT.)

Solution.

(a) From the time specs defined in the question, we have:

$$t_r \leq 1.8 \Longrightarrow \frac{1.8}{\omega_n} \leq 1.8 \Longrightarrow \omega_n \geq 1$$

$$t_s \leq 6.5 \Longrightarrow \frac{3}{\zeta\omega_n} \leq 6.5 \Longrightarrow \zeta\omega_n \geq \frac{6}{13}$$

The poles of the above second order system can be written as: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. To ensure the stability, we need $\zeta\omega_n > 0$. To ensure the above time specs, we need the real part of the poles $\text{Re}\{s_{1,2}\} = -\zeta\omega_n \leq -\frac{6}{13}$. The magnitude of the poles $|s_{1,2}| = \omega_n \geq 1$. The shaded area of Figure 2 shows the admissible pole locations.

(b) For this specific system, we have $\omega_n = \sqrt{3} > 1$ and $-\zeta\omega_n = -2 < \frac{6}{13}$. Therefore, this system satisfies the required specifications.

(c) The system can be written as the following ODE form:

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 3u(t).$$

Noticing that in this question we have non-zero initial conditions, performing the Laplace transform of the above ODE gives:

$$s^2Y(s) - s - 2 + 4sY(s) - 4 + 3Y(s) = 3U(s) \Longrightarrow Y(s) = \frac{3U(s) + s + 6}{s^2 + 4s + 3}.$$

Since we have $u(t) = 2 - 2e^{-6}, t \geq 0$, $U(s) = \frac{2}{s} - \frac{2}{s+1} = \frac{2}{s(s+1)}$. Therefore, we have

$$sY(s) = s\frac{3U(s) + s + 6}{s^2 + 4s + 3} = \frac{6 + s(s+1)(s+6)}{(s+1)^2(s+3)},$$

which has all strictly stable poles, hence FVT can be applied. Specifically, we have:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{6 + s(s+1)(s+6)}{(s+1)^2(s+3)} = 2.$$

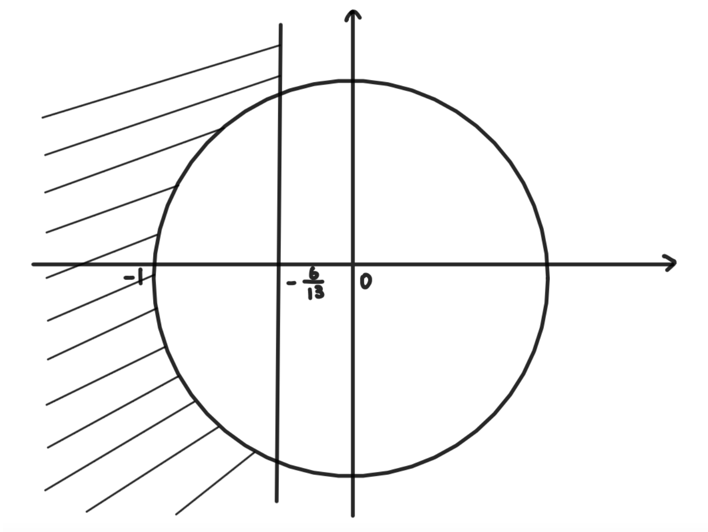


Figure 2: Admissible pole location for 2(b).

Problem 3.

(a) (10 points) Use the Routh-Hurwitz criterion to evaluate the stability of a system with the following transfer function:

$$T(s) = \frac{10}{s^5 + 3s^4 + 3s^3 + 7s^2 + 2s + 1}$$

Be sure to give the Routh-Hurwitz table as well as the number of unstable poles.

(b) (10 points) Now consider a different transfer function:

$$T(s) = \frac{10}{s^5 + 3s^4 + 5s^3 + 7s^2 + 2s + 1}$$

Use the Routh-Hurwitz criterion to evaluate the stability of the above system. Again, do the calculations by hand.

Solution.

(a) We have the following:

s^5	s^4	s^3	s^2	s^1	s^0
1	3	2/3	-1/2	3	1
3	7	5/3	1	0	0
2	1	0	0	0	0

Table 1: Routh table for part (a) - note the table is transposed

Since there is 2 sign changes in the first row, there are two unstable poles.

(b) We have the following:

s^5	s^4	s^3	s^2	s^1	s^0
1	3	8/3	41/8	47/41	1
5	7	5/3	1	0	0
2	1	0	0	0	0

Table 2: Routh table for part (b) - note the table is transposed

Since there are no sign changes in the first row, this system is stable.

Problem 4. Let $G(s) = \dfrac{15}{s^2 - 5s + 7}$. The output of G is denoted as y, and the reference is denoted as r. Suppose $e = r - y$. And the input to G is denoted as u.

(a) (5 points) What is the closed-loop transfer function from r to y if we use the PI controller $u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau$? Is it possible to choose (K_p, K_i) such that the closed-loop transfer function from r to y is stable? Justify your answers using Routh-Hurwitz criterion.

(b) (5 points) What is the closed-loop transfer function from r to y if we use the PID controller $u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \dot{e}(t)$?

(c) (5 points) How to choose the PID control gains (from (b)) such that the closed-loop transfer function from r to y has repeated poles at -4? (In other words, all the poles of the closed-loop transfer function from r to y are equal to -4.)

(d) (5 points) How to choose the PID control gains (from (b)) such that the closed-loop transfer function from r to y has repeated poles at -2? (In other words, all the poles of the closed-loop transfer function from r to y are equal to -2.)

(e) (5 points) Compare the controllers designed in (c) and (d). Which one has a faster unit step response (assuming zero initial conditions)? Which one can reject the sensor noise in a better way?

Solution.

(a) Given above $G(s)$ for a PI controller we have that $U(s) = K_p + K_i/s$ and the closed loop transfer function is given by $H(s) = \dfrac{U(s)G(s)}{1 + U(s)G(s)}$. Expanded we get:

$$H(s) = \frac{15\left(K_i + K_p s\right)}{15K_i + \left(15K_p + 7\right)s - 5s^2 + s^3}$$

We have no control over the s^2 term whose coefficient violates the necessary condition for stability. Therefore it is not possible to choose K_p, K_i to achieve stability.

(b) Now we have that $U(s) = K_p + K_i/s + K_d s$ and the closed loop transfer function is:

$$H(s) = \frac{15\left(K_i + K_p s + K_d s^2\right)}{15K_i + (7 + 15K_p)\, s + (15K_d - 5)\, s^2 + s^3}$$

(c) For all poles at -4 we need the following characteristic equation:

$$(s+4)^3 = s^3 + 12s^2 + 48s + 64$$

to be the same as the denominator of $H(s)$ above. Then comparing terms we get

$$15Ki = 64 \implies K_i = \frac{64}{15}$$
$$7 + 15K_p = 48 \implies K_p = \frac{41}{15}$$
$$15Kd - 5 = 12 \implies K_d = \frac{17}{15}$$

(d) By a similar procedure for all poles at -2, we need characteristic equation $s^3+6s^2+12s+8$. Therefore,

$$K_i = \frac{8}{15} \qquad K_p = \frac{1}{3} \qquad K_d = \frac{11}{15}$$

(e) The poles of the controller (c) are more negative than those of controller (d) and so the former will have a faster step response, see Figure 3. However, this comes at the cost of larger K_p, K_i, K_d gains (as seen above) which may also amplify noise. Therefore controller (d) will reject sensor noise better.

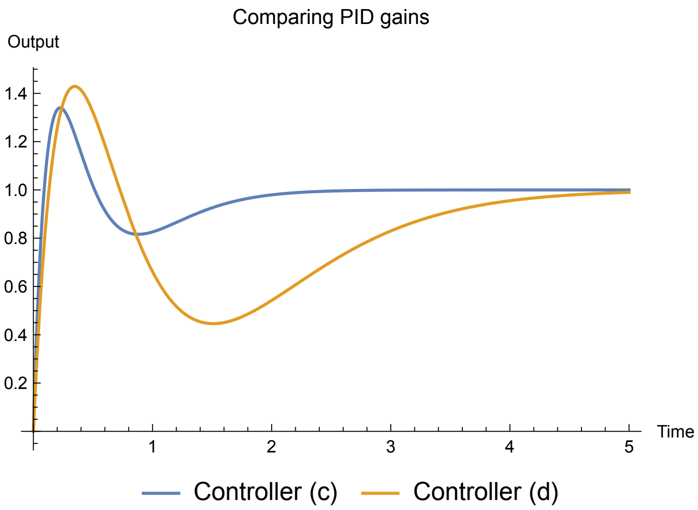


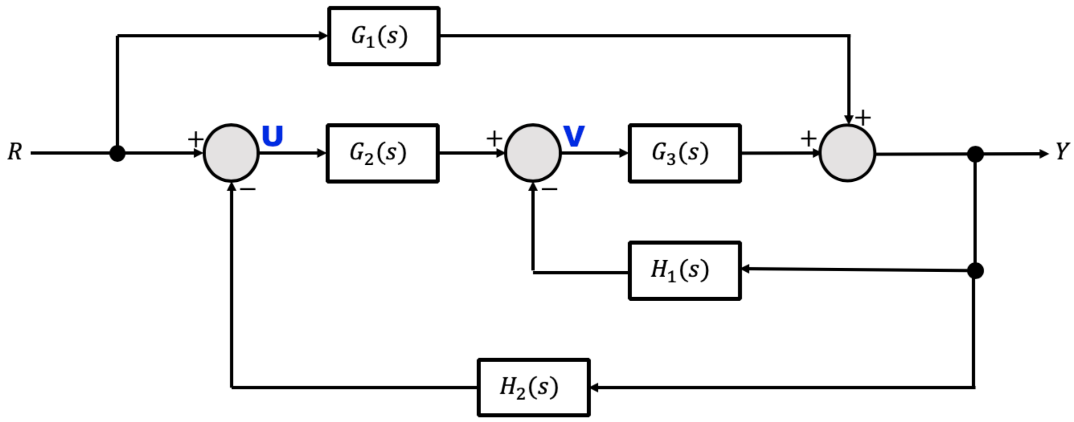
Figure 3: Comparing the step response of controllers,

Problem 5.

(a) (5 points) Determine the transfer function $\dfrac{Y(s)}{R(s)}$ from the block diagram below. (All G_i and H_i are single-input, single-output systems.)

(b) (5 points) Consider the following system:

$$G(s) = \frac{15808}{s^5 + 50s^4 + 837s^3 + 5040s^2 + 10336s + 16640}$$



Construct a first-order or second-order approximation from the dominant pole. Do you expect the dominant pole approximation to be accurate? Use `Matlab` (or another numerical tool) to plot the step responses for G(s) and your dominant pole approximation on a single figure.

Solution.

(a) Define the quantites denoted in the figure as follows:

$$\begin{aligned} U(s) &= R(s) - H_2(s)Y(s) \\ V(s) &= G_2(s)U(s) - H_1(s)(Y(s) \\ Y(s) &= G_3(s)V(s) + G_1(s)R(s) \end{aligned}$$

Then

$$\begin{aligned} Y(s) &= G_3(s)\left(G_2(s)U(s) - H_1(s)Y(s)\right) + G_1(s)R(s) \\ &= G_3(s)\left(G_2(s)\left(R(s) - H_2(s)Y(s)\right) - H_1(s)Y(s)\right) + G_1(s)R(s) \end{aligned}$$

Therefore,

$$\frac{Y(s)}{R(s)} = \frac{G_3(s)G_2(s) + G_1(s)}{1 + G_3(s)G_2(s)H_2(s) + G_3(s)H_1(s)}$$

(b) Factor the denominator as:

$$s^5 + 50s^4 + 837s^3 + 5040s^2 + 10336s + 16640 = (s+8)\left(s^2 + 2s + 5\right)\left(s^2 + 40s + 416\right)$$

yielding that the dominant poles $\{-1 \pm 2i\}$ belong to the second factor $\left(s^2 + 2s + 5\right)$. Then construct a second order approximation as

$$\hat{G}(s) = \frac{z_0}{(s^2 + 2s + 5)}$$

with $z_0 = \dfrac{19}{4}$ obtained by setting $\hat{G}(0) = G(0)$.

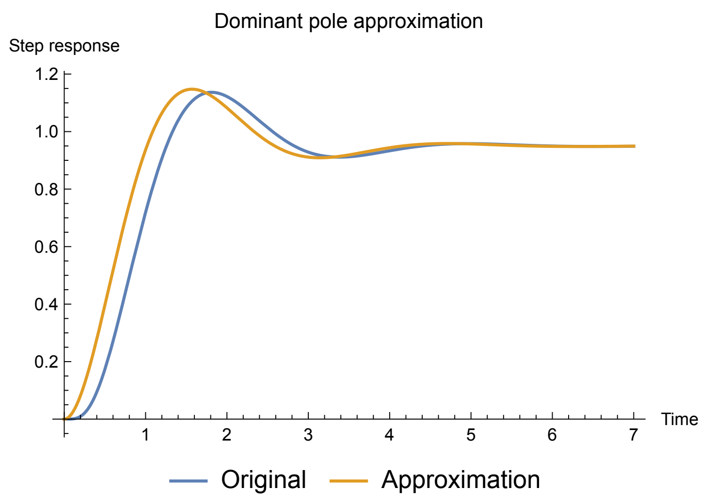


Figure 4: Step response of the system and its approximation