**Problem 1.**

(a) (10 points) Let $G(s)$ be a plant transfer function with 13 poles and 12 zeros. Suppose that all 12 zeros have negative real parts. Show that a large enough proportional feedback gain K (in standard negative unity feedback configuration) makes the closed-loop system stable.

(b) (10 points) What is a lag controller? Write down a transfer function of a lag controller. Be sure to explain how it differs from a lead controller. Give an intuitive explanation for the name "lag controller". Your explanation can rely either on the frequency response formula or on Bode plots.

**Solution.**

(a) We can answer this question using root-locus principles. We have 12 branches that go from poles to the LHP zeros. One more branch goes to negative infinity along the negative real axis. We can see this either by looking at the real axis part of the root-locus (to the left of all real axis poles and zeros) or from the rule for asymptotes:

$$\frac{180^\circ}{n-m} = 180^\circ$$

where in the question $n = m + 1$. Therefore for large $K$ the root-locus is in the left half plane.
**Note**: Attempts to solve this problem using the transfer function or Bode plots are pretty much useless; and will only accrue little partial credit.

(b) A lag controller is one whose transfer function takes the form

$$D(s) = \frac{s + z_{lag}}{s + p_{lag}}$$

with $z_{lag} > p_{lag}$ (which is the opposite with lead). The intuitive explanation for the name lag controller is that if the input is $\cos(\omega t)$ the output is $M\cos(\omega t + \phi)$ where

$$\phi = \angle \frac{j\omega + z_{lag}}{j\omega + p_{lag}} = \angle(jw + z_{lag}) - \angle(j\omega + p_{lag}) < 0$$

since $z_{lag} > p_{lag}$ which subtracts phase. Hence "lag". This can also be seen from the Bode plot which dips below zero.

**Problem 2.**

(a) (10 points) Consider the following system $G(s)$ and sinusoidal input:

$$\begin{aligned} &-3\dot{y}(t) - 2y(t) = 7u(t) \\ &u(t) = 6\cos(t+4) \\ &y(0) = 0 \end{aligned}$$

What is the magnitude and phase of $G(1j)$? Is the steady-state output bounded? If yes, what is it? Draw the Bode plots (both magnitude and phase) by hand. How large is the corner frequency?

(b) (10 points) Consider the following system $G(s)$ and sinusoidal input:

$$\begin{aligned} &\ddot{y}(t) + 0.1\dot{y}(t) + 4y(t) = \dot{u}(t) + 2u(t) \\ &u(t) = -\cos(2t) \\ &y(0) = 0 \end{aligned}$$

What is the magnitude and phase of $G(2j)$? Is the steady-state output bounded? If yes, what is it? Draw the Bode plots (both magnitude and phase) by hand.

(c) (10 points) Figure 1 shows an input $u(t)$ and the corresponding output $y(t)$ generated by a linear system $G(s)$. The input has the form $u(t) = A_0 \cos(\omega_0 t)$.
What are the values of $A_0$ and $\omega_0$ for the input signal? What is the magnitude $|G(j\omega_0)|$? What is the phase $\angle G(j\omega_0)$ in degrees? Based on Figure 1, what can we say about the Bode plot of $G(s)$?

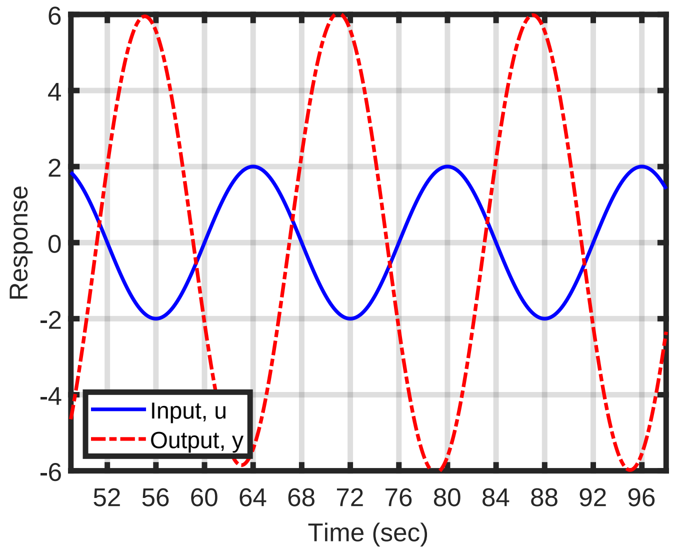


Figure 1: Input $u(t)$ and output response $y(t)$ for system $G(s)$.

**Solution.**
The key to solving this system is the fact that for a sinusoidal input $u(t) = A\cos(\omega t)$ an LTI system $G$ with zero initial conditions responds with a system output

$$y(t) = A\,|G(j\omega)|\cos(\omega t + \angle G(j\omega))$$

(a) We derive the transfer function as $G(s) = \dfrac{-7}{3s+2}$. This gives $|G(j)| = \dfrac{7\sqrt{13}}{13}$ and $\angle G(j) = \arctan\left(\dfrac{3}{-2}\right) = \pi - \arctan(3/2) = 123.69°$. The steady state output is bounded and can be readily obtained:

$$y_{ss}(t) = -\frac{42}{13}\left(2\cos(t+4) + 3\sin(t+4)\right) \tag{1}$$

The Bode plot is given in Figure 2 and the corner frequency $\omega_c = \dfrac{2}{3}$.

(b) Here the transfer function is given by

$$G(s) = \frac{s+2}{s^2 + s/10 + 4}$$

and so $|G(2j)| = 10\sqrt{2}$ and $\angle G(2j) = -\pi/4$. Again, the steady state output is bounded and given by:

$$y_{ss}(t) = -10\left(\cos(2t) + \sin(2t)\right) \tag{2}$$

The Bode plot is given below in Figure 3:

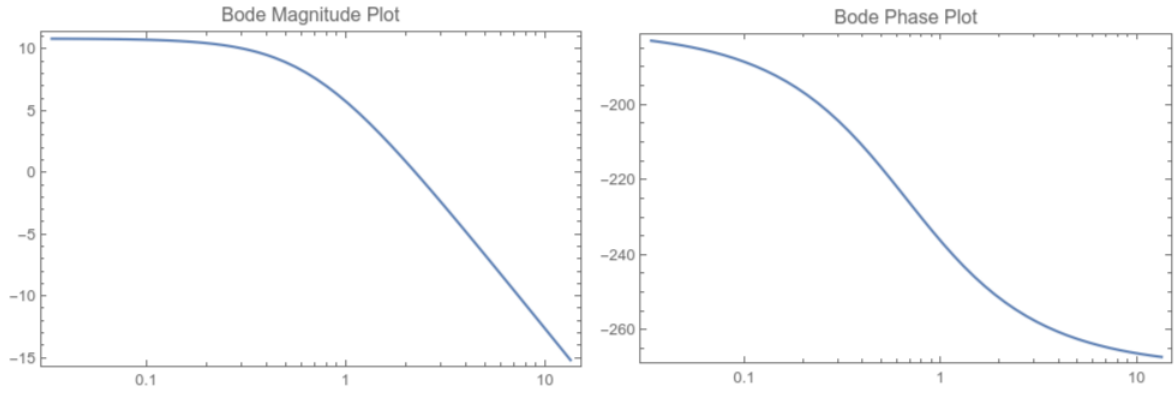


Figure 2: Bode plot for Part (a) of the question

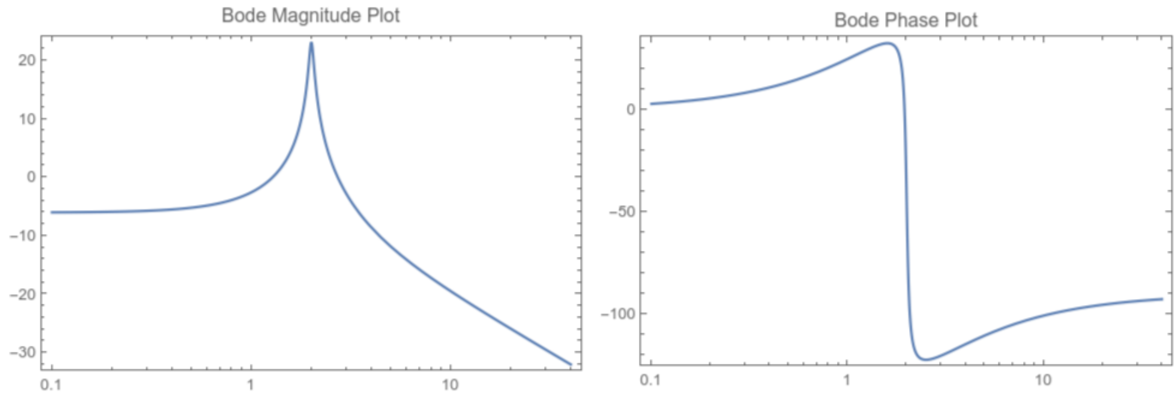


Figure 3: Bode plot for Part (b) of the question

(c) From the blue waveform $A_0 = 2$ and $\omega_0 = \frac{2\pi}{80-64} = \frac{\pi}{8}$. Comparing with the output wave, we get $|G(j\omega_0)| = \frac{6}{A_0} = 3$. Comparing their periods we have that $\angle G(j\omega_0) = -\frac{7}{16} \times 2\pi = -\frac{7\pi}{8} \approxeq -157.5°$. We cannot say much about the Bode plot apart from the fact that a point with this phase and magnitude exists on it.

**Remark:** As long the output was deemed bounded, points were given regardless of the fact whether Eqns (1) or (2) were derived.

**Problem 3.** (20 points) For the two transfer functions and gain values given below, use the Bode plot (generated by Matlab) to find the gain and phase margins:

(a) $K = 20$, and $G = \dfrac{1}{(s-1)(s+2)(s+6)}$

(b) $K = 2$, and $G = \dfrac{1}{(s+1)^3}$

For this problem, you are allowed to use MATLAB for drawing an accurate Bode plot. You should explain how you use the Bode plot to calculate the gain/phase margins.

**Solution.**
The phase margin can be found by noting down the frequency at which $|KG(j\omega)| = 1$ (or equivalently 0 db) on the Bode magnitude plot and then calculating how far the Bode phase plot is from the instability at $-180°$ at this frequency.

The gain margin can be found by noting down the frequency at which $\angle L(j\omega_0) = -180°$ on the Bode phase plot and then calculating $1/|L(j\omega_0)|$.

1. The phase margin is calculated to be 8.75°. The gain margin is [0.6, 2].

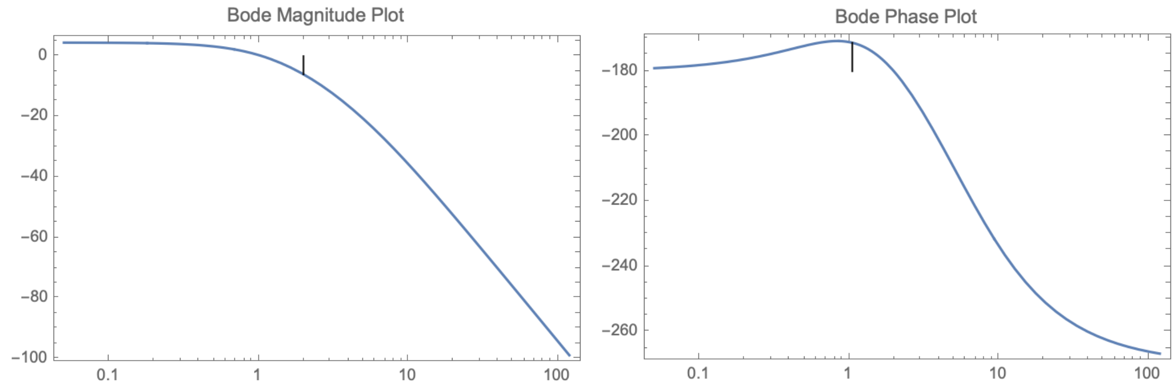


Figure 4: Bode plot for Part (a) of the question

2. The phase margin is calculated to be 67.59°. The gain margin is 4.

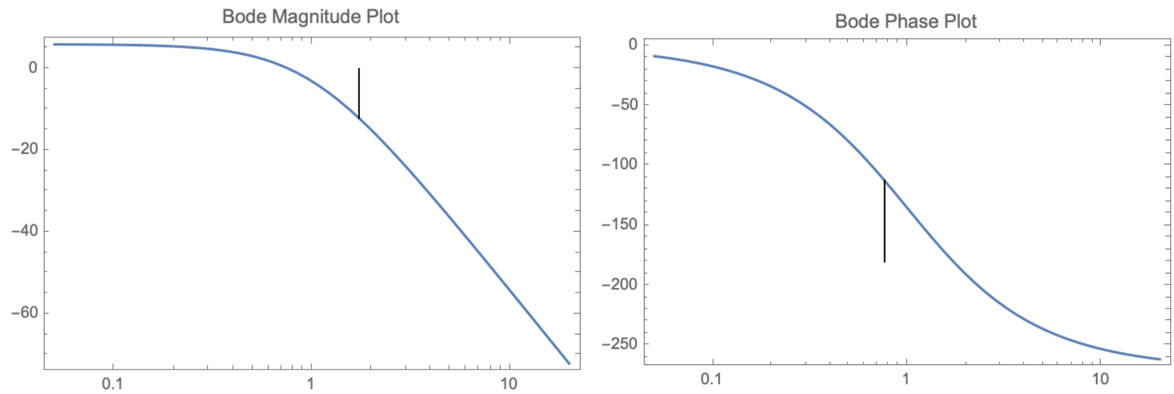


Figure 5: Bode plot for Part (b) of the question

**Problem 4.** For the following two loop transfer functions

(a) $L(s) = \dfrac{s+10}{2s+5}$.

(b) $L(s) = \dfrac{2}{(s-1)(s^2+2s+4)}$.

Draw the Nyquist plots by hand. Consider a standard closed-loop system with the loop transfer function given by $L(s)$. Apply the Nyquist stability theorem to predict the number of closed-loop poles of the feedback system in the RHP. Is the closed-loop system stable or unstable?

**Solution.**

(a) See Figure 6 for the hand drawn Bode plot. The magnitude decreases strictly from 2 to 0.5, while the phase starts at 0, decreases to a minimum of around -12 and then increases back to 0. This means the Nyquist plot hits the x or y axis only at 2 and 0.5 corresponding to $\omega = 0$ and $\omega = \infty$. So the Nyquist plot from 0 to $\infty$ can only be an arc from 2 to 0.5 below the x axis since phase is always negative here. This arc corresponds to the part of the Nyquist plot in Figure 7 marked by a single arrow. The Nyquist plot from $-\infty$ to 0 can be obtained by mirroring the part from 0 to $\infty$, and this portion is marked with double arrows in Figure 7.

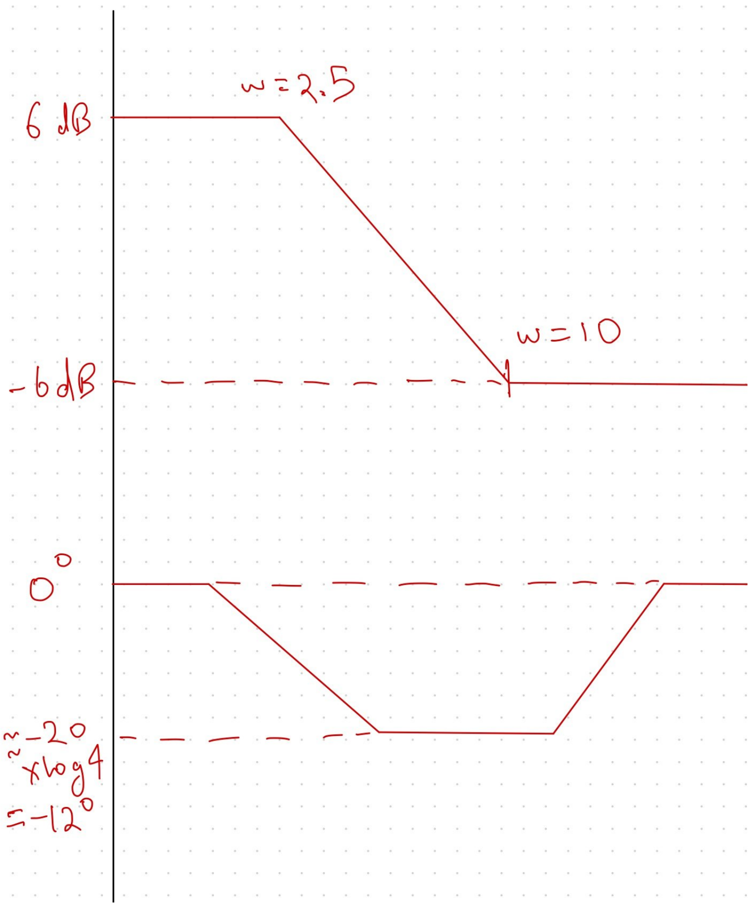


Figure 6: Bode plot of system 4a

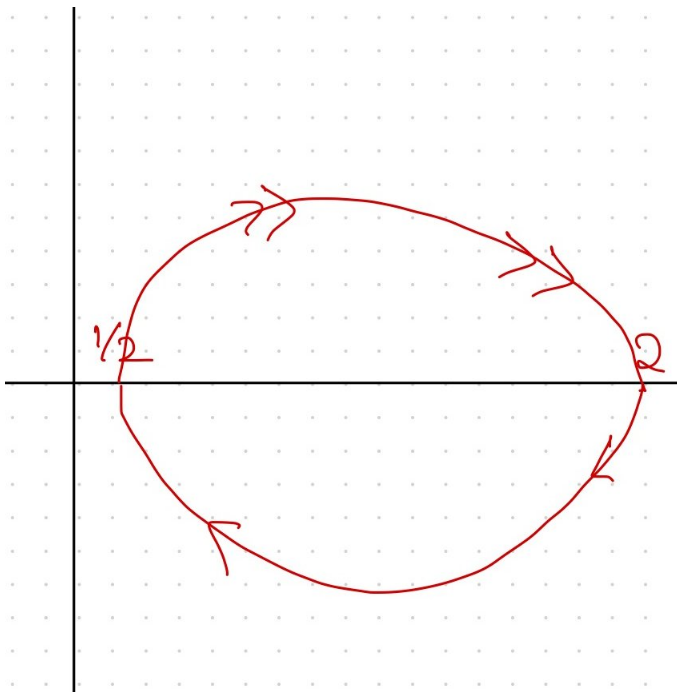


Figure 7: Nyquist plot of system 4a

Clearly, the number of windings around -1, $N = 0$ and $P = 0$. So $Z = 0$ and the closed loop is stable.

(b) See Figure 8 for the hand drawn Bode plot. The final phase plot is drawn in green. The magnitude decreases strictly from 0.5 to 0, while the phase starts at -180, increases for a while, and then decreases to -270. This means the Nyquist plot starts at -0.5 on the negative x-axis, goes above the x axis for a while, hits the x-axis back at a point between -0.5 and 0, goes below the x axis and finally approaches the origin tangent to the negative y-axis as $\omega$ goes to infinity. This path is marked with a single arrow in Figure 9. The rest of the nyquist plot can be obtained by mirroring, as indicated with double arrows in Figure 9.

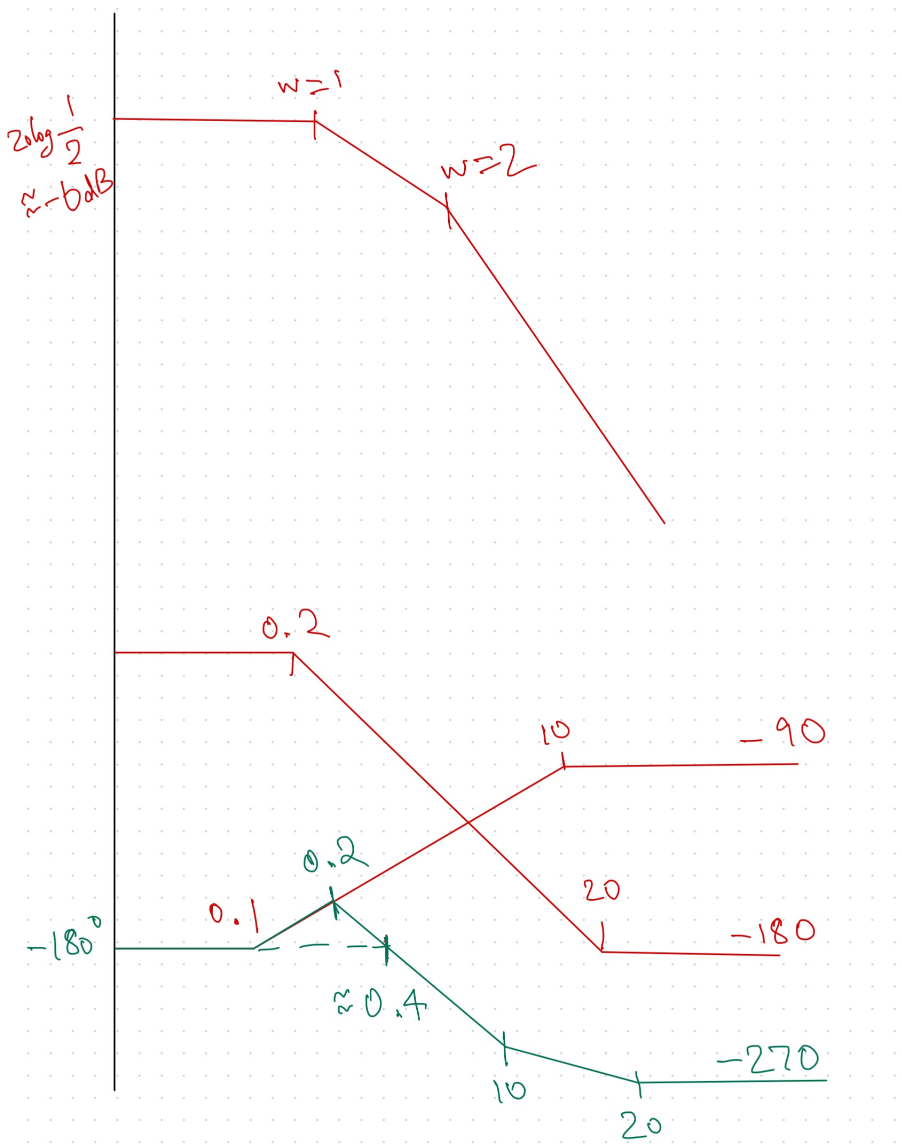


Figure 8: Bode plot of system 4a

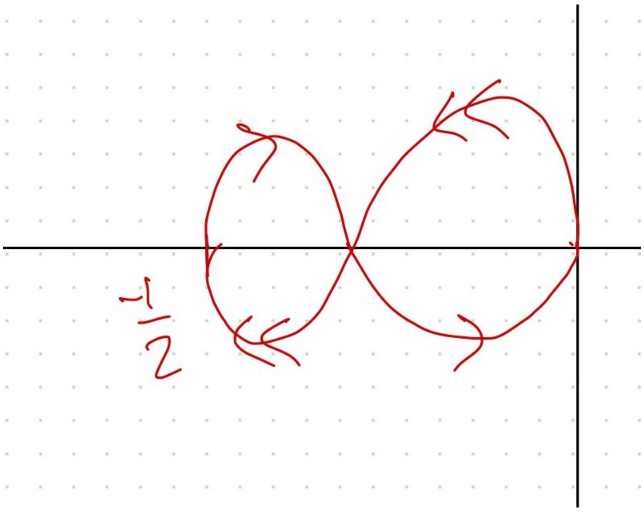


Figure 9: Nyquist plot of system 4a

Clearly, the number of windings around -1, $N = 0$ and $P = 1$. So $Z = 1$ and the closed loop is unstable.