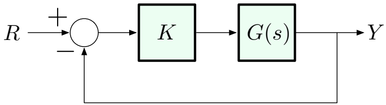
Plan of the Lecture

- Review: introduction to frequency-response design method
- Today's topic: Bode plots for three types of transfer functions

Goal: learn to analyze and sketch magnitude and phase plots of transfer functions written in Bode form (arbitrary products of three types of factors).

Reading: FPE, Section 6.1

Frequency-Response Design Method: Main Idea



Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior.

We will work with two types of plots for $KG(j\omega)$:

1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency ω (could have seen it earlier, in ECE 342)
2. Nyquist plots: $\text{Im}\big(KG(j\omega)\big)$ vs. $\text{Re}\big(K(j\omega)\big)$ [Cartesian plot in s-plane] as ω ranges from $-\infty$ to $+\infty$

Scale Convention for Bode Plots

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

Example:

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

$$\text{rewrite as} \quad \left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$

Three Types of Factors

Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where n is a positive or negative integer
2. $(j\omega\tau + 1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

In our example above,

$$KG(j\omega) = \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2 + \frac{j\omega}{2} + 1\right]}$$

$$= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}} \cdot \underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}} \cdot \underbrace{\left[\left(\frac{j\omega}{2}\right)^2 + \frac{j\omega}{2} + 1\right]^{-1}}_{\text{Type 3}}$$

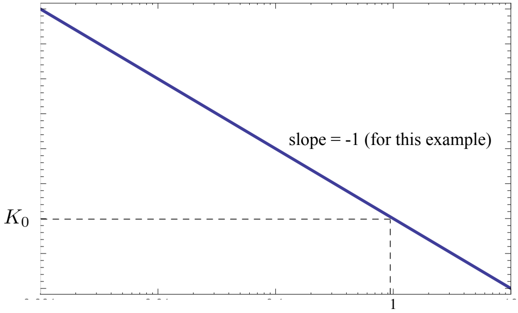
Now let's discuss Bode plots for factors of each type.

Type 1: $K_0(j\omega)^n$

Magnitude: $\log M = \log |K_0(j\omega)^n| = \log |K_0| + n \log \omega$

— as a function of $\log \omega$, this is a *line* of slope n passing through the value $\log |K_0|$ at $\omega = 1$

In our example, we had $K_0(j\omega)^{-1}$:

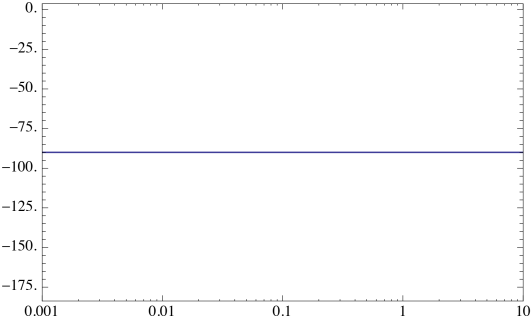


— this is called a low-frequency asymptote (will see why later)

Type 1: $K_0(j\omega)^n$

Phase: $\angle K_0(j\omega)^n = \angle(j\omega)^n = n\angle j\omega = n \cdot 90^\circ$
— this is a constant, independent of ω.

In our example, we had $K_0(j\omega)^{-1}$:

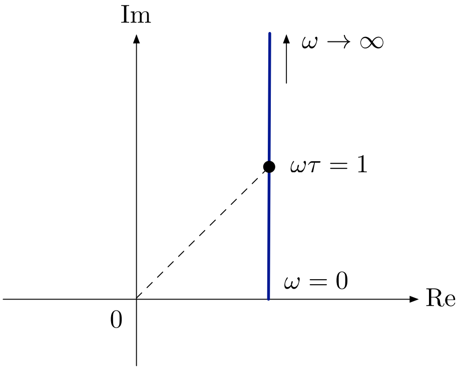


— here, the phase is -90° for all ω.

Type 2: $j\omega\tau + 1$

This is the case of a *stable real zero*.

To study $|j\omega\tau + 1|$ and $\angle(j\omega\tau + 1)$ as a function of ω, we will look at the *Nyquist plot*:



For $\omega\tau \ll 1,\ j\omega\tau + 1 \approx 1$

$\omega\tau \gg 1,\ j\omega\tau + 1 \approx j\omega\tau$

(like Type 1 with $K_0 = \tau, n = 1$)

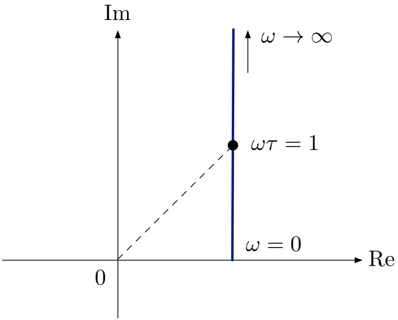
Transition:

$$\omega\tau = 1 \iff \omega = 1/\tau$$

— this is the breakpoint

Type 2: $j\omega\tau + 1$

Magnitude:



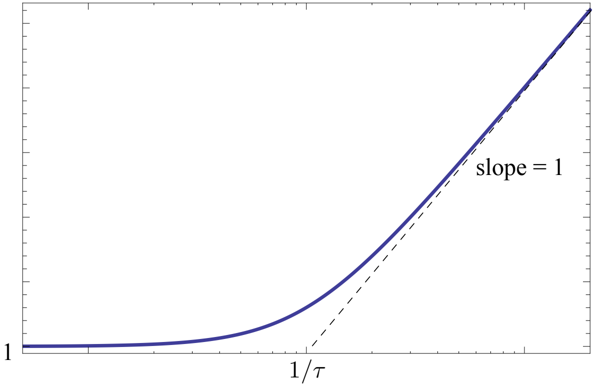
- For small ω (below break-point), $M \approx 1$ (horizontal line)
- For large ω (above break-point),

$$\begin{aligned}\log M \approx \log |j\omega\tau| &= \log \omega\tau \\ &= \log \tau + \log \omega\end{aligned}$$

  – a line of slope 1 passing through the point $(1/\tau, 1)$ (log-log scale)
- Careful: these are just *asymptotes* (the actual value of M at $\omega = 1/\tau$ is $\sqrt{2}$)

Type 2: $j\omega\tau + 1$

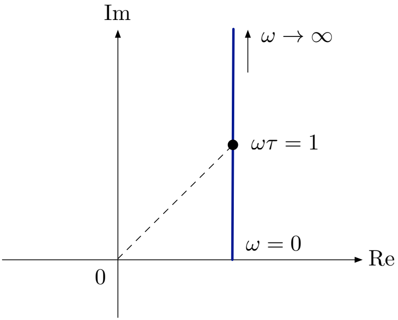
Magnitude plot:



For a stable real zero, the magnitude slope "steps up by 1" at the break-point.

Type 2: $j\omega\tau + 1$

Phase:



- For small ω (below break-point), $\phi \approx 0°$
- For large ω (above break-point),

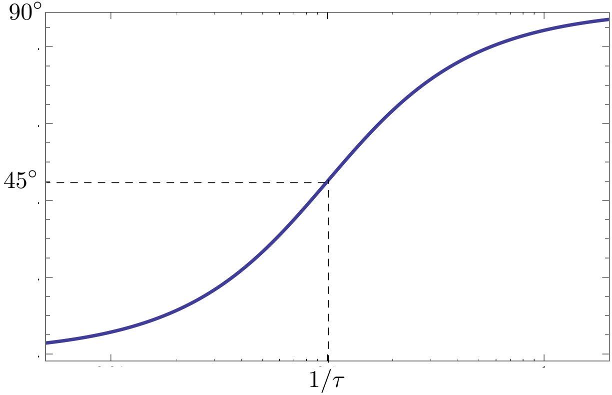
$$\begin{aligned} \phi &\approx \angle(j\omega\tau) \\ &= 90° \end{aligned}$$

- At break-point ($\omega\tau = 1$),

$$\begin{aligned} \phi &= \angle(j + 1) \\ &= 45° \end{aligned}$$

Type 2: $j\omega\tau + 1$

Phase plot:



For a stable real zero, the phase "steps up by 90°" as we go past the break-point.

Type 2: $(j\omega\tau + 1)^{-1}$

This is a stable real pole.

Magnitude:

$$\log \left| \frac{1}{j\omega\tau + 1} \right| = -\log |j\omega\tau + 1|$$

Phase:

$$\angle \frac{1}{j\omega\tau + 1} = -\angle (j\omega\tau + 1)$$

So the magnitude and phase plots for a stable real pole are the reflections of the corresponding plots for the stable real zero w.r.t. the horizontal axis:

- step down by 1 in magnitude slope
- step down by 90° in phase

Example: Type 1 and Type 2 Factors

$$KG(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

Convert to Bode form:

$$KG(j\omega) = \frac{2000 \cdot 0.5 \cdot \left(\frac{j\omega}{0.5}+1\right)}{10 \cdot 50 \cdot j\omega \left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

$$= \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5}+1\right) \cdot \frac{1}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

Example 1: Magnitude

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

Type 1 term:

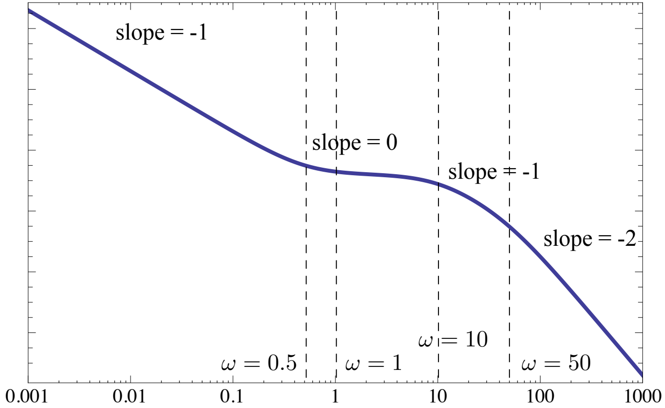
- $K_0 = 2, n = -1$ — it contributes a line of slope -1 passing through the point $(\omega = 1, M = 2)$.
- This is a low-frequency asymptote: for small ω, it gives very large values of M, while other terms for small ω are close to $M = 1$ (since $\log 1 = 0$).

Now we mark the break-points, from Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ slope steps up by 1
- $\omega = 10$ stable pole $\Rightarrow$ slope steps down by 1
- $\omega = 50$ stable pole $\Rightarrow$ slope steps down by 1

Example 1: Magnitude Plot

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



Example 1: Phase

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

Type 1 term:

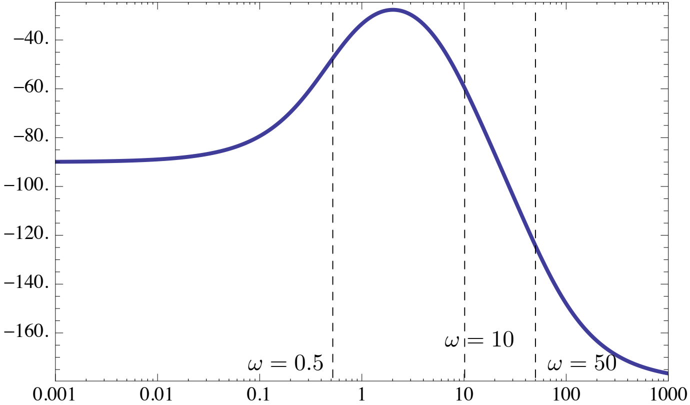
- $n = -1$ — phase starts at -90°

Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ phase up by 90° (by 45° at $\omega = 0.5$)
- $\omega = 10$ stable pole $\Rightarrow$ phase down by 90° (by 45° at $\omega = 10$)
- $\omega = 50$ stable pole $\Rightarrow$ phase down by 90° (by 45° at $\omega = 50$)

Example 1: Phase Plot

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



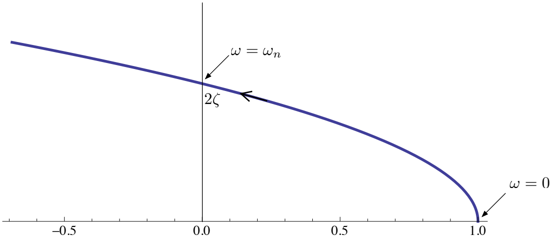
Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$

Stable complex zero — more difficult than Types 1 & 2.

First step — let's rewrite in Cartesian form:

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) + 2\zeta\frac{\omega}{\omega_n}j$$

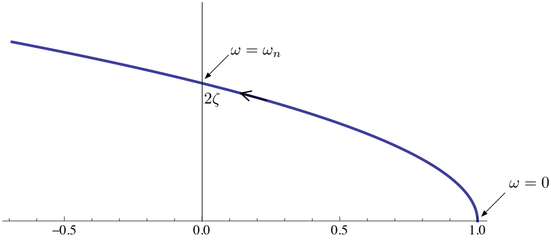
And here is the Nyquist plot, for $0 < \omega < \infty$:



$$(R(\omega), I(\omega))$$
$$= \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$

Nyquist plot, for $0 < \omega < \infty$:



$$(R(\omega), I(\omega)) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

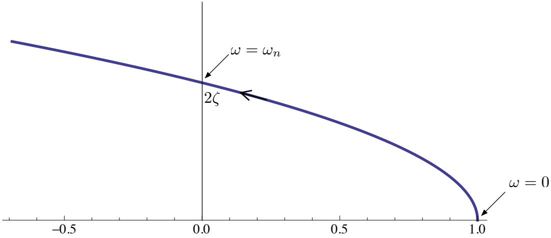
Some obvious points: $\omega = 0 \quad \to 1 + 0j$

$\omega = \omega_n \quad \to 0 + 2\zeta j$

What happens as $\omega \to \infty$?

- real part $\approx -(\omega/\omega_n)^2 \to -\infty$, quadratic in ω
- imaginary part $= 2\zeta(\omega/\omega_n) \to \infty$, linear in ω

Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$, Magnitude



Nyquist plot
$(0 < \omega < \infty)$

$$(R(\omega), I(\omega)) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

Magnitude:

- for $\omega \ll \omega_n$, $M \approx 1$ (horizontal line)
- for $\omega \gg \omega_n$, $M \approx \left(\frac{\omega}{\omega_n}\right)^2 \Rightarrow \log M \approx 2\log\omega - 2\log\omega_n$
  The asymptote is a line of slope 2 passing through the point $(\omega = \omega_n, M = 1)$

For a stable complex zero, the magnitude slope steps up by 2 as we go through the breakpoint.

Type 3: $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$

This is a stable complex pole.

Magnitude:

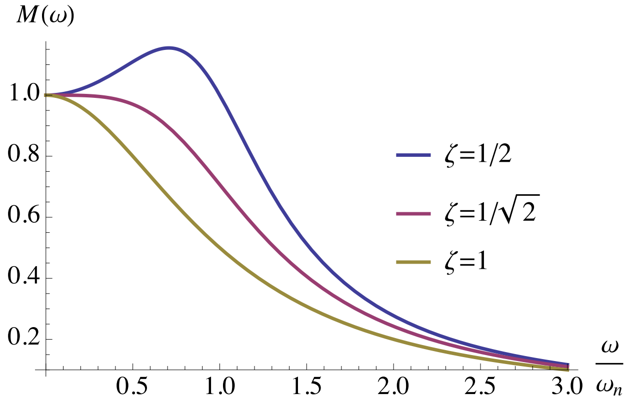
$$\log M = \log\left|\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}\right| = -\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right|$$

Phase:

$$\phi = \angle\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} = -\angle\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]$$

Type 3: Magnitude, Complex Pole Case

How does the magnitude plot look? Depends on the value of ζ:



The magnitude hits its peak value (for $\zeta < 1/\sqrt{2} \approx 0.707$) occurs when $\omega = \omega_r$, where

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} < \omega_n$$

Type 3: Magnitude

For small enough ζ (below $1/\sqrt{2}$), the magnitude of

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

has a resonant peak at the resonant frequency

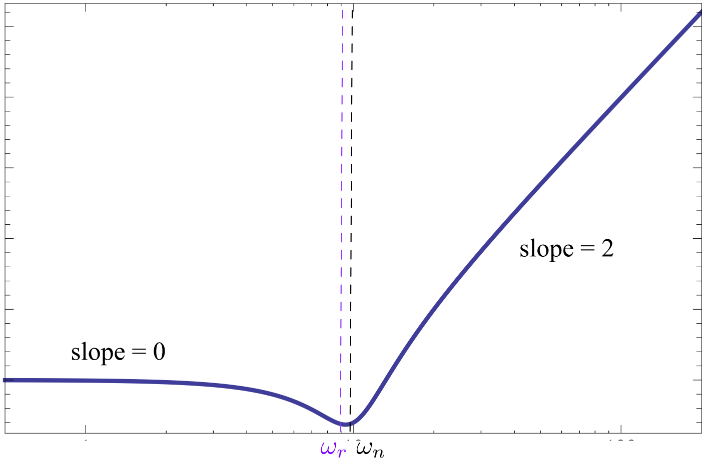
$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}.$$

Likewise, the magnitude of

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

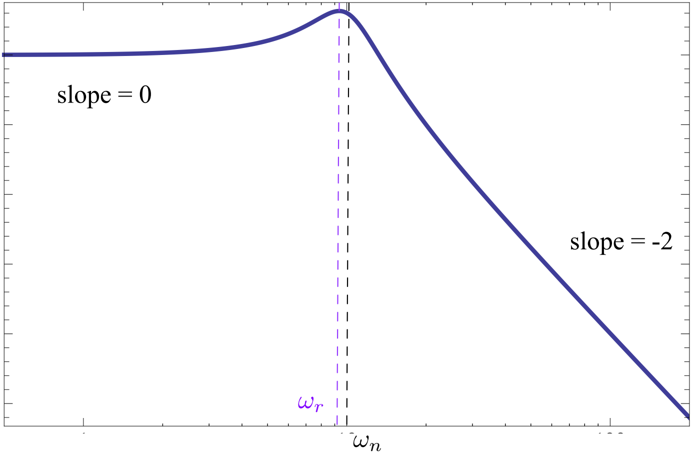
has a resonant dip at ω_r.

Type 3 Zero: Magnitude



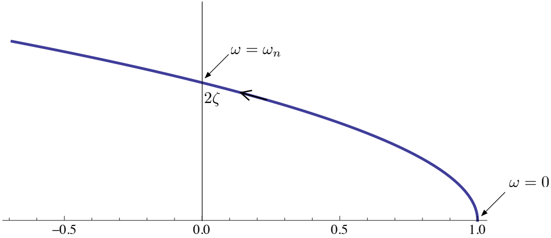
For a stable real zero, the magnitude slope “steps up by 2” at the break-point.

Type 3 Pole: Magnitude



For a stable real pole, the magnitude slope “steps down by 2” at the break-point.

Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$, Phase



Nyquist plot
$(0 < \omega < \infty)$

$$(R(\omega), I(\omega)) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

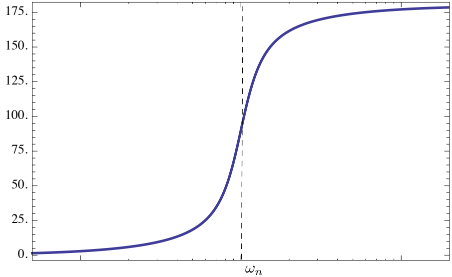
Phase:

- for $\omega \ll \omega_n$, $\phi \approx 0^\circ$ (real and positive)
- for $\omega = \omega_n$, $\phi = 90^\circ$ (Re $= 0$, Im > 0)
- for $\omega \gg \omega_n$, $\phi \approx 180^\circ$ (Re $\sim -\omega^2$, Im $\sim \omega$)

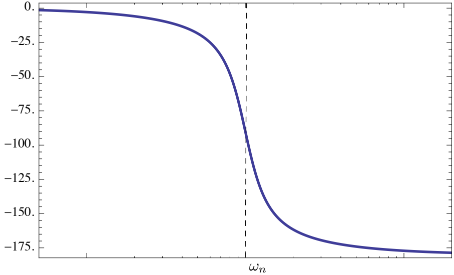
For a stable complex zero, the phase steps up by 180° as we go through the breakpoint; as $\zeta \to 0$, the transition through the break-point gets sharper, almost step-like.

For a pole, the phase is multiplied by -1.

Type 3: Phase



(stable complex zero — phase steps up by 180°)



(stable complex pole — phase steps down by 180°)

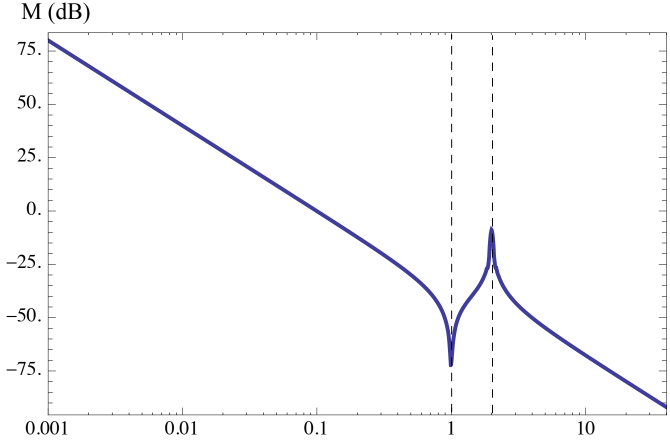
Example 2

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\dfrac{s^2}{4} + 0.02\dfrac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about magnitude?

- low-frequency term $\dfrac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — asymptote has slope $= -2$, passes through $(\omega = 1, M = 0.01)$
- complex zero with break-point at $\omega_n = 1$ and $\zeta = 0.005$ — slope up by 2; large resonant dip
- complex pole with break-point at $\omega_n = 2$ and $\zeta = 0.01$ — slope down by 2; large resonant peak

Example 2: Magnitude Plot



Example 2

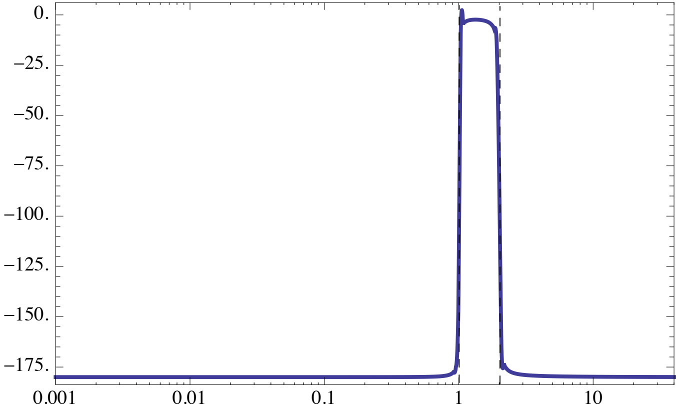
$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$

— already in Bode form

What can we tell about phase?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — phase starts at $n \times 90° = -180°$
- complex zero with break-point at $\omega_n = 1$ — phase up by 180°
- complex pole with break-point at $\omega_n = 2$ — phase down by 180°
- since ζ is small for both pole and zero, the transitions are very sharp

Example 2: Phase Plot



Unstable Zeros/Poles?

So far, we've only looked at transfer functions with stable poles and zeros (except perhaps at the origin). What about RHP?

Example: consider two transfer functions,

$$G_1(s) = \frac{s+1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{s-1}{s+5}$$

Note:

- G_1 has stable poles and zeros; G_2 has a RHP zero.
- Magnitude plots of G_1 and G_2 are the same —

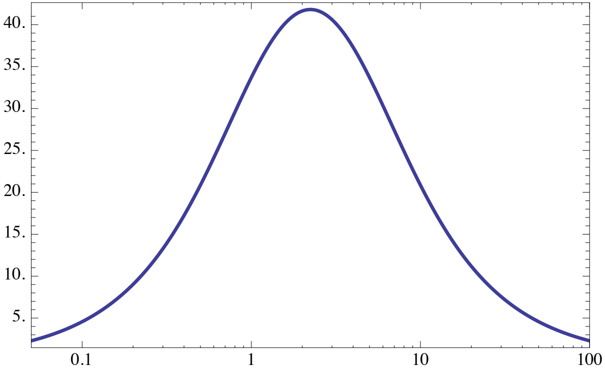
$$|G_1(j\omega)| = \left|\frac{j\omega+1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$
$$|G_2(j\omega)| = \left|\frac{j\omega-1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

- All the difference is in the phase plots!

Phase Plot for G_1

$$G_1(j\omega) = \frac{j\omega + 1}{j\omega + 5} = \frac{1}{5} \frac{j\omega + 1}{\frac{j\omega}{5} + 1}$$

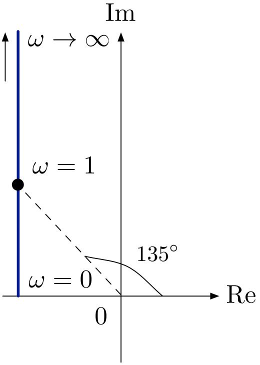
- Low-frequency term: $\frac{1}{5}(j\omega)^0$— $n = 0$, so phase starts at $0°$
- Break-points at $\omega_n = 1$ (phase goes up by $90°$) and at $\omega_n = 5$ (phase goes down by $90°$)



Phase Plot for G_2

$$G_2(j\omega) = \frac{j\omega - 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega - 1}{\frac{j\omega}{5} + 1}$$

Let's do a Nyqiust plot for $j\omega - 1$:

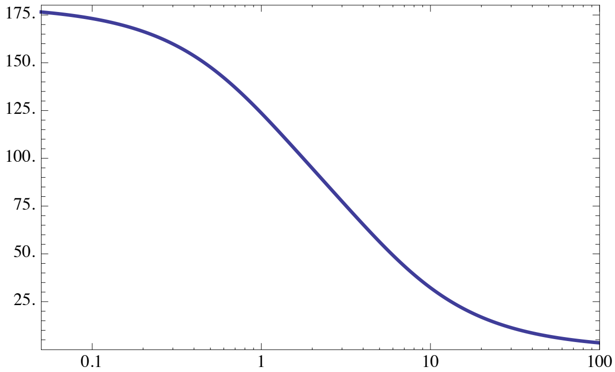


New type of behavior —

- $\omega \approx 0$: $\phi \approx 180^\circ$ (real and negative)
- $\omega \gg 1$: $\phi \approx 90^\circ$ (Re $= -1$, Im $= \omega \gg 1$)
- $\omega \approx 1$: $\phi \approx 135^\circ$

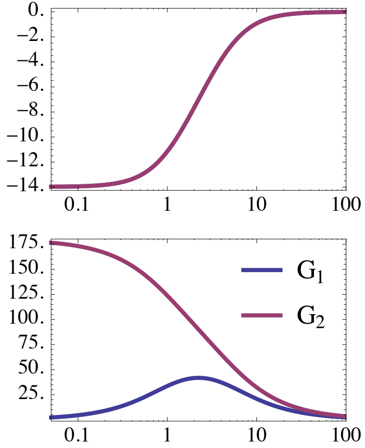
For a RHP zero, the phase starts out at 180° and goes down by 90° through the break-point (135° at break-point).

Phase Plot for G_2



For a RHP zero, the phase plot is similar to what we had for a LHP pole: goes down by 90° ... However, it starts at 180°, and not at 0°.

Minimum-Phase and Nonminimum-Phase Zeros



Among all transfer functions with the same magnitude plot, the one with only LHP zeros has the minimal net phase change as ω goes from 0 to ∞ — hence the term *minimum-phase* for LHP zeros.